

Galoisian Approach to integrability of Schrödinger Equation

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Dedicated to Jerry Kovacic, in memoriam.

ABSTRACT. In this paper, we examine the non-relativistic stationary Schrödinger equation from a differential Galois-theoretic perspective. The main algorithmic tools are pullbacks of second order ordinary linear differential operators, so as to achieve rational function coefficients (“algebrization”), and Kovacic’s algorithm for solving the resulting equations. In particular, we use this Galoisian approach to analyze Darboux transformations, Crum iterations and supersymmetric quantum mechanics. We obtain the ground states, eigenvalues, eigenfunctions, eigenstates and differential Galois groups of a large class of Schrödinger equations, e.g. those with exactly solvable and shape invariant potentials (the terms are defined within). Finally, we introduce a method for determining when exact solvability is possible.

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1. Introduction

In classical mechanics, the Hamiltonian corresponding to the energy (kinetic plus potential) is given by

$$H = \frac{\|\vec{p}\|^2}{2m} + U(\vec{x}), \quad \vec{p} = (p_1, \dots, p_n), \quad \vec{x} = (x_1, \dots, x_n),$$

and the corresponding (Hamiltonian) equations are $\dot{\vec{x}} = \frac{\partial H}{\partial \vec{p}}$ and $\dot{\vec{p}} = -\frac{\partial H}{\partial \vec{x}}$. In quantum mechanics, the momentum \vec{p} is replaced by $\vec{p} = -i\hbar\nabla$, the Hamiltonian operator by the Schrödinger (non-relativistic, stationary) operator

$$H = -\frac{\hbar^2}{2m}\nabla^2 + U(\vec{x})$$

and Hamilton's equations by the Schrödinger equation $H\Psi = E\Psi$.

The eigenfunction Ψ is known as the *wave function*, the eigenvalue E is the *energy level*, $U(\vec{x})$ is the *potential or potential energy*. Solutions (Ψ, E) of the Schrödinger equation are the *states* of the particle. In specific applications, the potentials will satisfy some conditions depending of physical restrictions such as barrier, scattering, etc., see [16, 27, 38, 42, 54].

The Schrödinger operator is known to be a *self-adjoint operator*: $H^\dagger = H$ when viewed within the context of a suitable complex and separable Hilbert space. As a consequence, the spectrum $\text{Spec}(H)$ of H must be real, bounded below, and be the disjoint union of the *point spectrum* $\text{Spec}_p(H)$ and the continuous spectrum $\text{Spec}_c(H)$ (see for example [5, 50, 61]). Wave functions belonging to the continuous spectrum may arise from barrier potentials and from periodic boundary conditions. The *transmission and reflection coefficients* are related with the barrier potentials, [27].

In this paper, we consider the one-dimensional Schrödinger equation with the normalization $\hbar = 2m = 1$, i.e.

$$H\Psi = E\Psi, \quad H = -\partial_z^2 + V(z),$$

where we may have $z = x$ (cartesian coordinate) or $z = r$ (radial coordinate). We let Ψ_n denote the wave function corresponding to an energy level $E = E_n$.

Our purpose is to investigate solutions and their classification from the viewpoint of differential Galois theory. We should point out that this viewpoint has already been explored in nearby theories, for example by Spiridonov in [60]. For sake of completeness, we now recall a few definitions related to quantum mechanics, and several useful results on the one-dimensional Schrödinger equation.

Bound States A solution Ψ_n is called a *bound state* when its norm is finite and the corresponding eigenvalue E_n belongs to the point spectrum of H , i.e.

$$E_n \in \text{Spec}_p(H), \quad \int |\Psi_n(x)|^2 dx < \infty, \quad n \in \mathbb{Z}_+.$$

Ground State and Excited States. Wave functions $\Psi_0, \Psi_1, \dots, \Psi_n, \dots$ of the bound states are numbered so that the corresponding energies are ordered $E_0 < E_1 < \dots < E_n < \dots$. The state (Ψ_0, E_0) with minimal energy is called the *ground state* and the remaining states $\Psi_1, \dots, \Psi_n, \dots$ are called the *excited states*.

Scattering States. A solution Ψ corresponding to the level energy E is called a *scattering state* when E belongs to the continuous spectrum of H and its norm is infinite.

To model a particle moving in one dimension, we use the classical one dimensional Schrödinger equation with Cartesian coordinate x . We could also consider motion in three dimension by reducing the associated Schrödinger equation to the one-dimensional *radial equation* (see [27, 38, 54]).

Darboux Transformation. The Darboux transformation is a process which, starting from a second order linear differential equation such as the Schrödinger equation, produces a family of second order differential equations with a similar shape. This vague formulation is made clear in the following theorem, taken from [20], which is slightly more general than what is known today as the Darboux transformation.

For any value of a constant m , let $y = y_m$ denote a solution of the equation

$$\partial_x^2 y + P \partial_x y + (Q - mR)y = 0.$$

Suppose, in addition, that we are given a solution θ of the equation

$$\partial_x^2 \theta + P \partial_x \theta + Q \theta = 0.$$

Then the function

$$u = \frac{\partial_x y - \frac{\partial_x \theta}{\theta} y}{\sqrt{R}},$$

is a solution of the equation

$$\partial_x^2 u + P \partial_x u + \left(\theta \sqrt{R} \partial_x \left(\frac{P}{\theta \sqrt{R}} \right) - \theta \sqrt{R} \partial_x^2 \left(\frac{1}{\theta \sqrt{R}} \right) - mR \right) u = 0,$$

when $m \neq 0$.

This *general Darboux transformation* $(y, \theta) \mapsto u$ preserves the shape of the differential equation and maps Q to $\theta \sqrt{R} \partial_x \left(\frac{P}{\theta \sqrt{R}} \right) - \theta \sqrt{R} \partial_x^2 \left(\frac{1}{\theta \sqrt{R}} \right)$. In [20, 21], Darboux considered the particular case $R = 1$ and $P = 0$, which is known today as the *Darboux transformation*. It is the following corollary of the above:

For any value of a constant m , let $y = y_m$ denote a solution of the equation

$$\partial_x^2 y = (f(x) + m)y.$$

Suppose, in addition, that we are given a solution θ of the equation $\partial_x^2 \theta = (f(x) + m_1)\theta$ for some fixed value m_1 (e.g $m_1 = 0$). Then the function

$$u = \partial_x y - \frac{\partial_x \theta}{\theta} y$$

is a solution of the equation

$$\partial_x^2 u = \left(\theta \partial_x^2 \left(\frac{1}{\theta} \right) - m_1 + m \right) u,$$

for $m \neq m_1$. Furthermore,

$$\theta \partial_x^2 \left(\frac{1}{\theta} \right) - m_1 = f(x) - 2 \partial_x \left(\frac{\partial_x \theta}{\theta} \right) = 2 \left(\frac{\partial_x \theta}{\theta} \right)^2 - f(x) - 2m_1.$$

Some examples can be found as exercises in the book of Ince [30, p. 132]. Darboux's transformation is related with Delsarte's transformation operators (named transmutations in [22]), which today are called *intertwiners* or *intertwining operators*. A good short survey about Darboux transformations can be found in [52].

Crum, inspired by the works of Liouville [39, 40] obtained an iterative generalization of Darboux's result appropriate for Sturm-Liouville systems: he proved that the Sturm-Liouville conditions are preserved under Darboux transformations, see [19]. Crum's result is presented in the following theorem. Recall that the Wronskian determinant Wr of k functions f_1, f_2, \dots, f_k is

$$Wr(f_1, \dots, f_k) = \det A, \quad A_{ij} = \partial_x^{i-1} f_j, \quad i, j = 1, 2, \dots, k.$$

Crum's Theorem. Let $\Psi_1, \Psi_2, \dots, \Psi_n$ be solutions of the Schrödinger equation $H\Psi = E\Psi$ for energy levels $E = E_1, E_2, \dots, E_n$ respectively. Letting

$$\Psi[n] = \frac{Wr(\Psi_1, \dots, \Psi_n, \Psi)}{Wr(\Psi_1, \dots, \Psi_n)}, \quad \text{and} \quad V[n] = V - 2\partial_x^2 \ln Wr(\Psi_1, \dots, \Psi_n),$$

we obtain the Schrödinger equation

$$H^{[n]}\Psi[n] = E\Psi[n], \quad \text{where} \quad H^{[n]} = -\partial_x^2 + V[n] \quad \text{for} \quad E \neq E_i, 1 \leq i \leq n$$

Crum's transformation coincides with Darboux's transformation when $n = 1$; as shown in [47], iterations of Darboux transformations coincide with the Crum iteration. Both formalisms allow us to obtain new families of Schrödinger equations while preserving the spectrum and the Sturm-Liouville conditions, see [42, 47]. We note that there are extensions of Crum's iteration connecting the Sturm-Liouville theory with orthogonal polynomial theory [37].

Supersymmetric quantum mechanical system. Witten introduced, in [72, §6], some models for which *dynamical breaking of supersymmetry* is possible.

A *supersymmetric quantum mechanical system* is described by a Hamiltonian \mathcal{H} together with several (self-adjoint) operators Q_i which commute with the hamiltonian ($[Q_i, \mathcal{H}] = 0$) and satisfy the anticommutation relations

$$\{Q_i, Q_j\} = \delta_{ij}\mathcal{H}, \quad \text{where} \quad \{Q_i, Q_j\} = Q_i Q_j + Q_j Q_i.$$

The simplest example of a supersymmetric quantum mechanical system arises in the case $n = 2$, which is considered in this paper. The wave function of $\mathcal{H}\Phi = E\Phi$ is then a two-component *Pauli spinor*,

$$\Phi(x) = \begin{pmatrix} \Psi_+(x) \\ \Psi_-(x) \end{pmatrix}.$$

The *supercharges* Q_i are defined as

$$Q_{\pm} = \frac{\sigma_1 p \pm \sigma_2 W(x)}{2}, \quad Q_+ = Q_1, \quad Q_- = Q_2, \quad p = -i\partial_x,$$

where the *superpotential* W is an arbitrary function of x and σ_i are the usual *Pauli spin matrices*. Using the expressions above we obtain \mathcal{H} :

$$\mathcal{H} = 2Q_-^2 = 2Q_+^2 = \frac{I_2 p^2 + I_2 W^2(x) + \sigma_3 \partial_x W(x)}{2}, \quad \text{with} \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The *supersymmetric partner Hamiltonians* H_{\pm} are given by

$$H_{\pm} = -\frac{1}{2}\partial_x^2 + V_{\pm}, \quad V_{\pm} = \left(\frac{W}{\sqrt{2}}\right)^2 \pm \frac{1}{\sqrt{2}}\partial_x \left(\frac{W}{\sqrt{2}}\right).$$

The potentials V_{\pm} are called *supersymmetric partner potentials* and are linked with the superpotential W through a Riccati equation. Then \mathcal{H} can be written as

$$\mathcal{H} = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix},$$

which leads us to the Schrödinger equations $H_+\Psi_+ = E\Psi_+$ and $H_-\Psi_- = E\Psi_-$. To solve $\mathcal{H}\Phi = E\Phi$ is equivalent to solve simultaneously $H_+\Psi_+ = E\Psi_+$ and $H_-\Psi_- = E\Psi_-$.

In [72, §6], V.B. Matveev and M. Salle recast Darboux transformations in this context; they interpret the Darboux theorem as a Darboux covariance of a Sturm-Liouville problem and prove the following result (see also [52, §5-6]).

The case $n = 2$ in Supersymmetric Quantum Mechanics is equivalent to a single Darboux transformation.

As a generalization of the method to solve the harmonic oscillator [28, 23], the ladder (raising and lowering) operators are defined as

$$A^+ = -\partial_x - \frac{\partial_x \Psi_0}{\Psi_0}, \quad A = \partial_x - \frac{\partial_x \Psi_0}{\Psi_0},$$

which are strongly related with the supercharges Q_{\pm} in Witten's formalism. We have,

$$A\Psi_0 = 0, \quad A^+A = H_-, \quad AA^+ = H_+ = -\partial_x^2 + V_+(x), \quad \text{where}$$

$$V_+(x) = V_-(x) - 2\partial_x \left(\frac{\partial_x \Psi_0}{\Psi_0} \right) = -V_-(x) + 2 \left(\frac{\partial_x \Psi_0}{\Psi_0} \right)^2.$$

The supersymmetric partner potentials V_+ and V_- have the same energy levels, except for $E_0^{(-)} = 0$. In terms of the superpotential $W(x)$, the operators A and A^+ are given by

$$A^+ = -\partial_x + W(x), \quad A = \partial_x + W(x).$$

Similarly, the supersymmetric partner potentials $V_{\pm}(x)$ and the superpotential $W(x)$ satisfy:

$$\frac{V_+(x) + V_-(x)}{2} = W^2(x), \quad [A, A^+] = 2\partial_x W(x).$$

Shape invariance. Gendenshtein, in his remarkable paper [28], introduced the concept of *shape invariance*, which is a property of families of potentials with respect to their parameters. Given a family $a = (a_n)_n$ of parameters, the shape invariance condition can be expressed as :

$$V_{n+1}(x; a_n) = V_n(x; a_{n+1}) + R(a_n), \quad V_- = V_0, \quad V_+ = V_1,$$

where R is a remainder, which does not depends on x . When this relation holds, we say that $V = V_- = V_0$ is a *shape invariant potential* (with respect to the given family or transformation), see also [52, 16, 23].

Gendenshtein Theorem. *Consider the Schrödinger equation $H\Psi = E_n\Psi$, where $V = V_- = V_0$ is a shape invariant potential. If we fix the first level of energy $E_0 = 0$, then the excited spectrum and the wave functions are given respectively by*

$$E_n = \sum_{k=2}^{n+1} R(a_k), \quad \Psi_n^{(-)}(x; a_1) = \prod_{k=1}^n A^+(x; a_k) \Psi_0^{(-)}(x; a_{n+1}).$$

The structure of the rest of the paper is as follows. In section 2 we introduce the necessary concepts and results of the Differential Galois Theory of linear ordinary differential equations. In section 3 we introduce some terminology about solvability and we obtain some results about the invariance of the Galois group with respect to Darboux transformations. Section 4 is devoted to the study of the rational potentials and section 5 to transcendental potentials by means of the algebrization method.

2. Differential Galois Theory

In this section, we recall some differential Galois material to study the one-dimensional Schrödinger equation,

$$\partial_x^2 \Psi = (V - \lambda)\Psi$$

where V is some function and λ is a constant (the spectral parameter). There are many references on differential Galois theory (or “Picard-Vessiot” theory), for example [11, 18, 43, 65].

2.1. Picard-Vessiot Galois theory. We establish an algebraic model for functions and the corresponding Galois theory.

Differential Fields. Let K be a commutative field of characteristic zero. A *derivation* of K is a map $\partial_x : K \rightarrow K$ satisfying $\partial_x(a + b) = \partial_x a + \partial_x b$ and $\partial_x(ab) = \partial_x a \cdot b + a \cdot \partial_x b$ for all $a, b \in K$. We then say that (K, ∂_x) (or just K , when there is no ambiguity) is a *differential field* with the derivation ∂_x . We assume that K contains an element x such that $\partial_x(x) = 1$. Let \mathcal{C} denote the field of constants of K :

$$\mathcal{C} = \{c \in K \mid \partial_x c = 0\}.$$

It is also of characteristic zero and will be assumed to be algebraically closed. Throughout this paper, the *coefficient field* for a differential equation (e.g the Schrödinger equation $\partial_x^2 \Psi = (V - \lambda)\Psi$) is defined as the smallest differential field containing all the coefficients of the equation.

We will mostly analyze second order linear homogeneous differential equations, i.e equations of the form

$$\mathcal{L} := \partial_x^2 y + a \partial_x y + by = 0, \quad a, b \in K.$$

so the rest of the theory will be explained in this context.

Picard-Vessiot Extension. Let L be a differential field containing K (a differential extension of K). We say that L is a *Picard-Vessiot* extension of K for \mathcal{L} if there exist two linearly independent $y_1, y_2 \in L$ solutions of \mathcal{L} such that $L = K\langle y_1, y_2 \rangle$ (i.e $L = K(y_1, y_2, \partial_x y_1, \partial_x y_2)$) and L and K has the same field of

constants \mathcal{C} .

In what follows, we choose a Picard-Vessiot extension and the term “solution of \mathcal{L} ” will mean “solution of \mathcal{L} in L ”. So any solution of \mathcal{L} is a linear combination (over \mathcal{C}) of y_1 and y_2 .

Differential Galois Groups A K -automorphism σ of the Picard-Vessiot extension L is called a differential automorphism if it leaves K fixed and commutes with the derivation. This means that $\sigma(\partial_x a) = \partial_x(\sigma(a))$ for all $a \in L$ and $\forall a \in K$, $\sigma(a) = a$.

The group of all differential automorphisms of L over K is called the *differential Galois group* of L over K and is denoted by $\text{DGal}(L/K)$.

Given $\sigma \in \text{DGal}(L/K)$, we see that $\{\sigma y_1, \sigma y_2\}$ are also solutions of \mathcal{L} . Hence there exists a matrix

$$A_\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}(2, \mathbb{C}),$$

such that

$$\sigma\left(\begin{pmatrix} y_1 & y_2 \end{pmatrix}\right) = \begin{pmatrix} \sigma(y_1) & \sigma(y_2) \end{pmatrix} = \begin{pmatrix} y_1 & y_2 \end{pmatrix} A_\sigma.$$

As σ commutes with the derivation, this extends naturally to an action on a fundamental solution matrix of the companion first order system associated with \mathcal{L} .

$$\sigma\left(\begin{pmatrix} y_1 & y_2 \\ \partial_x y_1 & \partial_x y_2 \end{pmatrix}\right) = \begin{pmatrix} \sigma(y_1) & \sigma(y_2) \\ \sigma(\partial_x y_1) & \sigma(\partial_x y_2) \end{pmatrix} = \begin{pmatrix} y_1 & y_2 \\ \partial_x y_1 & \partial_x y_2 \end{pmatrix} A_\sigma.$$

This defines a faithful representation $\text{DGal}(L/K) \rightarrow \text{GL}(2, \mathbb{C})$ and it is possible to consider $\text{DGal}(L/K)$ as a subgroup of $\text{GL}(2, \mathbb{C})$. It depends on the choice of the fundamental system $\{y_1, y_2\}$, but only up to conjugacy.

Recall that an algebraic group G is an algebraic manifold endowed with a group structure. Let $\text{GL}(n, \mathbb{C})$ denote, as usual, the set of invertible $n \times n$ matrices with entries in \mathbb{C} (and $\text{SL}(n, \mathbb{C})$ be the set of matrices with determinant equal to 1). A linear algebraic group will be a subgroup of $\text{GL}(n, \mathbb{C})$ equipped with a structure of algebraic group. One of the fundamental results of the Picard-Vessiot theory is the following theorem (see [33, 35]).

The differential Galois group $\text{DGal}(L/K)$ is an algebraic subgroup of $\text{GL}(2, \mathbb{C})$. In fact, the differential Galois group measures the algebraic relations between the solutions (and their derivatives). It is sometimes viewed as the object which should tell “what algebra sees of the dynamics of the solutions”.

In an algebraic group G , the largest connected algebraic subgroup of G containing the identity, noted G° , is a normal subgroup of finite index. It is often called the connected component of the identity. If $G = G^\circ$ then G is a *connected group*.

When G° satisfies some property, we say that G virtually satisfies this property. For example, virtually solvability of G means solvability of G° and virtual abelianity of

G means abelianity of G^0 (see [71]).

Lie-Kolchin Theorem. *Let $G \subseteq \mathrm{GL}(2, \mathbb{C})$ be a virtually solvable group. Then G^0 is triangularizable, i.e it is conjugate to a subgroup of upper triangular matrices.*

2.2. Algebraic Subgroups of $\mathrm{SL}(2, \mathbb{C})$. We recall below some examples of subgroups of $\mathrm{SL}(2, \mathbb{C})$ for later use.

Reducible subgroups These are the groups which leave a non-trivial subspace of V invariant. They are classified in two categories.

Diagonal groups: the identity group: $\{e\} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$, the n -roots: $\mathbb{G}^{[n]} = \left\{ \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix}, c^n = 1 \right\}$, the multiplicative group: $\mathbb{G}_m = \left\{ \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix}, c \in \mathbb{C}^* \right\}$

Triangular groups: the additive group: $\mathbb{G}_a = \left\{ \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}, d \in \mathbb{C} \right\}$, the n -quasi-roots: $\mathbb{G}^{\{n\}} = \left\{ \begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix}, c^n = 1, d \in \mathbb{C} \right\}$, the Borel group: $\mathbb{B} = \mathbb{C}^* \ltimes \mathbb{C} = \left\{ \begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix}, c \in \mathbb{C}^*, d \in \mathbb{C} \right\}$

Irreducible subgroups The infinite dihedral group (also called meta-abelian group):

$\mathbb{D}_\infty = \left\{ \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix}, c \in \mathbb{C}^* \right\} \cup \left\{ \begin{pmatrix} 0 & d \\ -d^{-1} & 0 \end{pmatrix}, d \in \mathbb{C}^* \right\}$ and its finite subgroups \mathbb{D}_{2n} (where c and d spans the n -th roots of unity).

There are also three other finite irreducible (primitive) groups: the tetrahedral group $A_4^{\mathrm{SL}_2}$ of order 24, the octahedral group $S_4^{\mathrm{SL}_2}$ of order 48, and the icosahedral group $A_5^{\mathrm{SL}_2}$ of order 120.

Integrability. We say that the linear differential equation \mathcal{L} is (Liouville) *integrable* if the Picard-Vessiot extension $L \supset K$ is obtained as a tower of differential fields $K = L_0 \subset L_1 \subset \cdots \subset L_m = L$ such that $L_i = L_{i-1}(\eta)$ for $i = 1, \dots, m$, where either

- (1) η is *algebraic* over L_{i-1} , that is η satisfies a polynomial equation with coefficients in L_{i-1} .
- (2) η is *primitive* over L_{i-1} , that is $\partial_x \eta \in L_{i-1}$.
- (3) η is *exponential* over L_{i-1} , that is $\partial_x \eta / \eta \in L_{i-1}$.

We remark that the usual terminology in differential algebra for integrable equations is that the corresponding Picard-Vessiot extensions are called *Liouvillian*. The following theorem is due to Kolchin.

The equation \mathcal{L} is integrable if and only if $\mathrm{DGal}(L/K)$ is virtually solvable.

2.3. Eigenrings. Eigenrings are a tool, introduced by M.F. Singer, to study the decomposability of systems or operators. A general reference is [6] (and references therein).

Eigenrings of Systems. Consider the system $\partial_x X = -AX$, denoted by $[A]$, where $A \in \text{GL}(2, K)$. The eigenring of the system $[A]$, denoted by $\mathcal{E}(A)$, is the set of 2×2 matrices P in K satisfying

$$\partial_x P = PA - AP.$$

Eigenrings of Operators. Let \mathfrak{L} be a second order differential operator, i.e $\mathcal{L} := \mathfrak{L}(y) = 0$. Denote $V(\mathfrak{L})$ as the solution space of \mathcal{L} . The *eigenring* of \mathfrak{L} , denoted by $\mathcal{E}(\mathfrak{L})$, is the set of all operators \mathfrak{R} of order less than that of \mathfrak{L} for which $\mathfrak{R}(V(\mathfrak{L}))$ is a subset of $V(\mathfrak{L})$, that is $\mathfrak{L}\mathfrak{R} = \mathfrak{S}\mathfrak{L}$, where \mathfrak{S} is also an operator. For more details see [59, 6, 66, 67, 68].

We study an equation $\mathfrak{L}(y) = 0$ with $\mathfrak{L} = \partial_x^2 + p\partial_x + q$; this is equivalent to the system $[A]$, where A is given by

$$A = \begin{pmatrix} 0 & -1 \\ q & p \end{pmatrix} \quad p, q \in K.$$

Relations between the system approach and the operator approach are given in the following simple lemmas .

Lemma. Let $\mathfrak{L} = \partial_x^2 + p\partial_x + q$, $A = \begin{pmatrix} 0 & -1 \\ q & p \end{pmatrix}$ and $P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then:

- (1) If $P \in \mathcal{E}(A)$, then $\mathfrak{R} = a + b\partial_x \in \mathcal{E}(\mathfrak{L})$.
- (2) If $\mathfrak{R} = a + b\partial_x \in \mathcal{E}(\mathfrak{L})$, then $P \in \mathcal{E}(A)$, where P is given by

$$P = \begin{pmatrix} a & b \\ \partial_x a - bq & a + \partial_x b - bp \end{pmatrix}.$$

- (3) $1 \leq \dim_{\mathbb{C}} \mathcal{E}(\mathfrak{L}) \leq 4$.
- (4) $P \in \text{GL}(2, K) \Leftrightarrow \frac{\partial_x a}{a} - \frac{a}{b} + p \neq \frac{\partial_x b}{b} - \frac{b}{a}q$.

Lemma. Assume that $\mathfrak{L} = \partial_x^2 + b$, where $b \in K$. The following statements hold:

- (1) If $\dim_{\mathbb{C}} \mathcal{E}(\mathfrak{L}) = 1$, then either the differential Galois group is irreducible (\mathbb{D}_{∞} or primitive), or indecomposable ($G \subseteq \mathbb{B}$, G non-diagonal).
- (2) If $\dim_{\mathbb{C}} \mathcal{E}(\mathfrak{L}) = 2$, then either the differential Galois group is the additive group or it is contained in the multiplicative group (but is not the identity group).
- (3) If $\dim_{\mathbb{C}} \mathcal{E}(\mathfrak{L}) = 4$, then the differential Galois group is the identity group. In this case we have 2 independent solutions ζ_1 and ζ_2 in which ζ_1^2, ζ_2^2 and $\zeta_1\zeta_2$ are elements of the differential field K , i.e. the solutions of $\mathcal{L}^{\otimes 2}$ belong to the differential field K .

2.4. Some useful special functions. Some standard special functions will be used below.

The Gauss hypergeometric equation. It is a particular case of *Riemann's equation* (see [32, 48, 34, 41, 24]) and is given by

$$\partial_x^2 y + \frac{(\gamma - (\kappa + \beta + 1)x)}{x(1-x)} \partial_x y - \frac{\kappa\beta}{x(1-x)} y = 0.$$

The cases when it is solvable can be found in the Kimura table [34] (or via Kovacic's algorithm, see appendix).

The confluent hypergeometric equations (Kummer, Whittaker). It is a degenerate form of the hypergeometric differential equation where two of the three regular singularities merge into an irregular singularity. For example, making “1 tend to ∞ ” in a suitable way, the hypergeometric equation may transform to two classical forms:

- *Kummer’s form*

$$\partial_x^2 y + \frac{c-x}{x} \partial_x y - \frac{a}{x} y = 0$$

- *Whittaker’s form*

$$\partial_x^2 y = \left(\frac{1}{4} - \frac{\kappa}{x} + \frac{4\mu^2 - 1}{4x^2} \right) y$$

where the parameters of the two equations are linked by $\kappa = \frac{c}{2} - a$ and $\mu = \frac{c}{2} - \frac{1}{2}$. The Galoisian structure of these equations has been deeply studied in [41, 24]. The following theorem is due to Martinet & Ramis, [41].

The Whittaker’s differential equation is integrable if and only if either, $\kappa + \mu \in \frac{1}{2} + \mathbb{N}$, or $\kappa - \mu \in \frac{1}{2} + \mathbb{N}$, or $-\kappa + \mu \in \frac{1}{2} + \mathbb{N}$, or $-\kappa - \mu \in \frac{1}{2} + \mathbb{N}$.

The Bessel equation. It is a particular case of the confluent hypergeometric equation and is given by

$$\partial_x^2 y + \frac{1}{x} \partial_x y + \frac{x^2 - n^2}{x^2} y = 0.$$

Under a suitable transformation, the reduced form of the Bessel’s equation is a particular case of the Whittaker’s equation. Thus, we can obtain the following well known result, see [35, p. 417] and see also [36, 43].

The Bessel’s differential equation is integrable if and only if $n \in \frac{1}{2} + \mathbb{Z}$.

By double confluence of the hypergeometric equation, that is making “0 and 1 tend to ∞ ” in a suitable way, one gets the *parabolic cylinder equation* (also known as *Weber’s equation*):

$$\partial_x^2 y = \left(\frac{1}{4} x^2 - \frac{1}{2} - n \right) y,$$

which is integrable if and only if $n \in \mathbb{Z}$, see [36, 24]. Under suitable transformations one can get the Rehm’s form of the Weber’s equation:

$$\partial_x^2 y = (ax^2 + 2bx + c) y, \quad a \neq 0,$$

so that $\frac{b^2 - c}{a}$ is an odd integer.

The hypergeometric equation, including confluences, is a particular case of the differential equation

$$\partial_x^2 y + \frac{L}{Q} \partial_x y + \frac{\lambda}{Q} y, \quad \lambda \in \mathbb{C}, \quad L = a_0 + a_1 x, \quad Q = b_0 + b_1 x + b_2 x^2.$$

We mention this because the *classical orthogonal polynomials* and *Bessel polynomials* are solutions of such an equation (see [15, 31, 46]). We remark that integrability

conditions and solutions of differential equations with solutions orthogonal polynomials, including Bessel polynomials, can be obtained applying Kovacic's algorithm. Similarly, one can apply Kovacic's algorithm to obtain the same results given by Kimura [34] and Martinet & Ramis [41]. Also we recall that Duval & Loday-Richaud applied Kovacic's algorithm in [24] to some families of special functions, including several of the above.

3. Notions of Algebraically Solvable and Quasi-Solvable Potentials

The main object of our Galoisian analysis is the one-dimensional Schrödinger equation, written as

$$(3.1) \quad \mathcal{L}_\lambda := H\Psi = \lambda\Psi, \quad H = -\partial_x^2 + V(x), \quad V \in K,$$

where K is a differential field (with \mathbb{C} as field of constants). We are interested in the integrability (in Galoisian sense) of equation (3.1). Denote by L_λ the Picard-Vessiot extension of K for \mathcal{L}_λ . The differential Galois group of \mathcal{L}_λ is $\mathrm{DGal}(L_\lambda/K)$.

3.1. Algebraic Spectrum and Algebraic Solvability. Let $\Lambda \subseteq \mathbb{C}$ denote the set of eigenvalues λ such that equation (3.1) is integrable in Galoisian sense. The set Λ will be called *the algebraic spectrum* (or alternatively *the Liouvillian spectral set*) of H . We remark that Λ can be \emptyset , i.e., $\mathrm{DGal}(L_\lambda/K) = \mathrm{SL}(2, \mathbb{C}) \forall \lambda \in \mathbb{C}$. On the other hand, the classification of subgroups of $SL_2(\mathbb{C})$ shows that, if $\lambda_0 \in \Lambda$, then $(\mathrm{DGal}(L_{\lambda_0}/K))^0 \subseteq \mathbb{B}$.

We let Λ_+ be the set $\{\lambda \in \Lambda \cap \mathbb{R} : \lambda \geq 0\}$ and Λ_- be the set $\{\lambda \in \Lambda \cap \mathbb{R} : \lambda \leq 0\}$.

Definition 3.1 (Algebraically Solvable and Quasi-Solvable Potentials). We say that the potential $V(x) \in K$ is:

- an *algebraically solvable potential* when Λ is an infinite set, or
- an *algebraically quasi-solvable potential* when Λ is a non-empty finite set, or
- an *algebraically non-solvable potential* when $\Lambda = \emptyset$.

When $\mathrm{Card}(\Lambda) = 1$, we say that $V(x) \in K$ is a *trivial* algebraically quasi-solvable potential.

Examples. Let $K = \mathbb{C}(x)$.

- (1) If $V(x) = x$ (Airy equation), then $\Lambda = \emptyset$, $V(x)$ is algebraically non-solvable, see [33, 36].
- (2) If $V(x) = 0$, then $\Lambda = \mathbb{C}$, i.e., $V(x)$ is algebraically solvable. Furthermore,

$$\mathrm{DGal}(L_0/K) = \{e\}, \quad \text{and} \quad \mathrm{DGal}(L_\lambda/K) = \mathbb{G}_m \text{ for } \lambda \neq 0.$$

- (3) If $V(x) = \frac{x^2}{4} + \frac{1}{2}$, then $\Lambda = \{n : n \in \mathbb{Z}\}$, $V(x)$ is algebraically solvable. This example corresponds to the Weber's equation given previously.
- (4) If $V(x) = x^4 - 2x$, then $\Lambda = \{0\}$ and $V(x)$ is a trivial algebraically quasi-solvable potential (see next section for tools to prove this).

We are interested in the spectrum (analytic spectrum) of the algebraically solvable and quasi-solvable potentials, that is, $\mathrm{Spec}(H) \cap \Lambda \neq \emptyset$.

Potentials for which $\text{Spec}(H) \cap \Lambda$ is an infinite set are called *solvable* (or *exactly solvable*) *potentials* (see Natanzon [45]) in the physics litterature. Similarly, potentials for which $\text{Spec}(H) \cap \Lambda$ is a finite set are usually called *quasi-exactly solvable* (or *quasi-solvable*) *potentials* (Turbine [63], Bender & Dunne [8], Bender & Boettcher [7], Saad et al. [53], Gibbons & Vesselov [29]).

3.2. Changing the Base Field. In what follows, we will make transformations which may change the base field; typically, when K is not $\mathbb{C}(x)$, we will look for transformations to another differential with rational coefficients. We now collect facts on how this affects the differential Galois group and its solvability.

Definition 3.2. Let \mathcal{L} and $\tilde{\mathcal{L}}$ be a pair of linear differential equations defined over differential fields K and \tilde{K} respectively, with Picard-Vessiot extensions L and \tilde{L} . Let φ be a transformation mapping K to \tilde{K} and \mathcal{L} to $\tilde{\mathcal{L}}$ (so we may also choose φ to map L to \tilde{L}). We say that:

- (1) φ is an *isogaloisian transformation* if

$$\text{DGal}(L/K) = \text{DGal}(\tilde{L}/\tilde{K}).$$

When $\tilde{L} = L$ and $\tilde{K} = K$, we'll say that φ is a *strong isogaloisian transformation*.

- (2) φ is a *virtually isogaloisian transformation* if

$$(\text{DGal}(L/K))^0 = (\text{DGal}(\tilde{L}/\tilde{K}))^0.$$

Remark 3.3. The eigenring s of two operators \mathfrak{L} and $\tilde{\mathfrak{L}}$ are preserved under isogaloisian transformations.

Proposition 3.4. *Consider the differential equations*

$$\mathcal{L} := \partial_x^2 y + a \partial_x y + by = 0, \quad \tilde{\mathcal{L}} := \partial_x^2 \zeta = r \zeta, \quad a, b, r \in K.$$

Let $\kappa \in \mathbb{Q}$, $f \in K$, $a = 2\kappa \partial_x(\ln f)$ and φ be the transformation such that $\mathcal{L} \mapsto \tilde{\mathcal{L}}$. The following statements holds:

- (1) φ is a *strong isogaloisian transformation* for $\kappa \in \mathbb{Z}$.
(2) φ is a *virtually strong isogaloisian transformation* for $\kappa \in \mathbb{Q} \setminus \mathbb{Z}$.

PROOF. Let $\mathcal{B} = \{y_1, y_2\}$ be a basis of solutions of \mathcal{L} and L be the corresponding Picard-Vessiot extension of K ; similarly, let $\mathcal{B}' = \{\zeta_1, \zeta_2\}$ be a basis of solutions of $\tilde{\mathcal{L}}$ if a Picard-Vessiot extension \tilde{L} . With the change of dependent variable $y = \zeta e^{-\frac{1}{2} \int a}$ we obtain $r = a^2/4 + \partial_x a/2 - b$ and $\tilde{K} = K$. Thus, the relationship between L and \tilde{L} depends on a :

- (1) If $\kappa = n \in \mathbb{Z}$, then $\mathcal{B}' = \{f^n y_1, f^n y_2\}$ which means that $L = \tilde{L}$ and φ is strong isogaloisian.
(2) If $\kappa = \frac{n}{m}$, with $\gcd(n, m) = 1$, $\frac{n}{m} \notin \mathbb{Z}$, then $\mathcal{B}' = \{f^{\frac{n}{m}} y_1, f^{\frac{n}{m}} y_2\}$ which means that \tilde{L} is either an algebraic extension of degree at most m of L , and φ is virtually strong isogaloisian, or $L = \tilde{L}$ when $f^{\frac{n}{m}} \in K$ which means that φ is strong isogaloisian.

□

Remark 3.5. The transformation φ in proposition 3.4 is not injective, there are a lot of differential equations \mathcal{L} that are transformed in the same differential equation $\tilde{\mathcal{L}}$. They are called projectively equivalent.

The following is an easy consequence of the proposition:

Corollary 3.6. *Let \mathcal{L} be the differential equation*

$$\partial_x(a\partial_x y) = (\lambda b - \mu)y, \quad a, b \in K, \quad \lambda, \mu \in \mathbb{C}$$

where $L, \tilde{L}, \tilde{\mathcal{L}}$ and φ are given as in proposition 3.4. Then either \tilde{L} is a quadratic extension of L (which means that φ is virtually strong isogaloisian) or $\tilde{L} = L$ (this is when $a^{\frac{1}{2}} \in K$) which means that φ is strong isogaloisian.

4. Galoisian Aspects of Darboux Transformations and Shape Invariance

Here we present a Galoisian approach to the Darboux transformation, the Crum iteration and shape invariant potentials.

Recall that $Wr(y_1, \dots, y_n)$ denotes the Wronskian

$$Wr(y_1, \dots, y_n) = \begin{vmatrix} y_1 & \cdots & y_n \\ \vdots & & \vdots \\ \partial_x^{n-1} y_1 & \cdots & \partial_x^{n-1} y_n \end{vmatrix}.$$

The Darboux transformation will be abbreviated by DT and its n -th iteration by DT_n ; similarly, the abbreviation CI_n will refer to the Crum iteration.

As above, K denotes the coefficient field of the studied linear differential equation, i.e. the smallest differential containing the coefficients of the equation. We mostly consider equations over the base field $K = \mathbb{C}(x)$.

4.1. Darboux Transformations. The following result comes from the original Darboux transformation (see [20, 21]), but rewritten in a way which is adapted to our Galoisian analysis.

Theorem 4.1 (Darboux Transformation). *Consider the Schrödinger operators $H_{\pm} = \partial_x^2 + V_{\pm}(x)$, with $V_{\pm}(x) \in K$, and assume that the spectrum satisfies $\Lambda \neq \emptyset$. Let \mathcal{L}_{λ} (resp. $\tilde{\mathcal{L}}_{\lambda}$) denote the Schrödinger equation $H_{-}\Psi^{(-)} = \lambda\Psi^{(-)}$ (resp. $H_{+}\Psi^{(+)} = \lambda\Psi^{(+)}$).*

Let DT be the transformation such that $\mathcal{L} \mapsto \tilde{\mathcal{L}}$, $V_{-} \mapsto V_{+}$, $\Psi^{(-)} \mapsto \Psi^{(+)}$. Then the following statements holds:

- i): $DT(V_{-}) = V_{+} = \Psi_{\lambda_1}^{(-)} \partial_x^2 \left(\frac{1}{\Psi_{\lambda_1}^{(-)}} \right) + \lambda_1 = V_{-} - 2\partial_x^2(\ln \Psi_{\lambda_1}^{(-)})$,
 $DT(\Psi_{\lambda_1}^{(-)}) = \Psi_{\lambda_1}^{(+)} = \frac{1}{\Psi_{\lambda_1}^{(-)}}$, where $\Psi_{\lambda_1}^{(-)}$ is a particular solution of \mathcal{L}_{λ_1} ,
 $\lambda_1 \in \Lambda$.
- ii): $DT(\Psi^{(-)}) = \Psi^{(+)} = \partial_x \Psi^{(-)} - \partial_x(\ln \Psi_{\lambda_1}^{(-)})\Psi^{(-)} = \frac{W(\Psi_{\lambda_1}^{(-)}, \Psi^{(-)})}{W(\Psi_{\lambda_1}^{(-)})}$, $\lambda \neq \lambda_1$,
 where $\Psi^{(-)} = \Psi_{\lambda}^{(-)}$ is the general solution of \mathcal{L}_{λ} for $\lambda \in \Lambda \setminus \{\lambda_1\}$ and
 $\Psi^{(+)} = \Psi_{\lambda}^{(+)}$ is the general solution of $\tilde{\mathcal{L}}_{\lambda}$ also for $\lambda \in \Lambda \setminus \{\lambda_1\}$.

PROOF. We treat each item separately.

i): By hypothesis, we have $\text{Card}(\Lambda) \geq 1$, $\lambda_1 \in \Lambda$,

$H_- \Psi_{\lambda_1}^{(-)} = \lambda_1 \Psi_{\lambda_1}^{(-)}$, and setting $\tilde{H}_{\pm} = H_{\pm} - \lambda_1$, $\tilde{V}_{\pm} = V_{\pm} - \lambda_1$ we obtain $\tilde{H}_- \Psi_{\lambda_1}^{(-)} = 0$. Thus, the superpotential $W = -\partial_x \ln \Psi_{\lambda_1}^{(-)}$, depending of λ_1 , exists and satisfies

$$(-\partial_x + W)(\partial_x + W) \Psi_{\lambda_1}^{(-)} = \tilde{H}_- \Psi_{\lambda_1}^{(-)} = 0.$$

Now, developing both sides of the expression, we have $\tilde{V}_- = -\partial_x W + W^2$. So, interchanging operators we have

$$(\partial_x + W)(-\partial_x + W) \Psi_{\lambda_1}^{(+)} = \tilde{H}_+ \Psi_{\lambda_1}^{(+)} = 0,$$

so that developing both sides of the expression we have $\tilde{V}_+ = \partial_x W + W^2$ which lead us to $W = \partial_x \ln \Psi_{\lambda_1}^{(+)}$. Thus, we obtain

$$\text{DT}(\Psi_{\lambda_1}^{(-)}) = \Psi_{\lambda_1}^{(+)} = \frac{1}{\Psi_{\lambda_1}^{(-)}}.$$

On another hand, as $\tilde{V}_+ - \tilde{V}_- = V_+ - V_- = 2\partial_x W$, $\tilde{V}_+ + \tilde{V}_- = V_+ + V_- - 2\lambda_1 = 2W^2$, we obtain $\text{DT}(V_-) = V_+$ which is given by

$$V_- - 2\partial_x \left(\frac{\partial_x \Psi_{\lambda_1}^{(-)}}{\Psi_{\lambda_1}^{(-)}} \right) = 2 \left(\frac{\partial_x \Psi_{\lambda_1}^{(-)}}{\Psi_{\lambda_1}^{(-)}} \right)^2 - V_- + 2\lambda_1 = \Psi_{\lambda_1}^{(-)} \partial_x^2 \left(\frac{1}{\Psi_{\lambda_1}^{(-)}} \right) + \lambda_1.$$

ii): Suppose that $\text{Card}(\Lambda) > 1$, $\lambda \neq \lambda_1$ and $\Psi^{(\pm)} = \Psi_{\lambda}^{(\pm)}$ satisfying

$$H_- \Psi^{(-)} = \lambda \Psi^{(-)}, \quad \lambda \neq \lambda_1,$$

applying again the raising and lowering operators we have

$$(-\partial_x + W)(\partial_x + W) \Psi^{(-)} \neq 0, \quad (\partial_x + W)(-\partial_x + W) \Psi^{(+)} \neq 0,$$

so that

$$(-\partial_x + W)(\partial_x + W) \Psi^{(-)} = (-\partial_x + W) \Psi^{(+)},$$

we obtain

$$\begin{aligned} \Psi^{(+)} &= (\partial_x + W) \Psi^{(-)} = \partial_x \Psi^{(-)} + W \Psi^{(-)} \\ &= \partial_x \Psi^{(-)} - \frac{\partial_x \Psi_{\lambda_1}^{(-)}}{\Psi_{\lambda_1}^{(-)}} \Psi^{(-)} = \frac{W(\Psi_{\lambda_1}^{(-)}, \Psi^{(-)})}{W(\Psi_{\lambda_1}^{(-)})}. \end{aligned}$$

Now, we will show that effectively $\Psi^{(+)}$ satisfy $H_+ \Psi^{(+)} = \lambda \Psi^{(+)}$ for $\lambda \neq \lambda_1$. Let suppose that

$$\Psi^{(+)} = \partial_x \Psi^{(-)} - \Psi^{(-)} \frac{\partial_x \Psi_{\lambda_1}^{(-)}}{\Psi_{\lambda_1}^{(-)}},$$

we can see that

$$\partial_x^2 \Psi^{(-)} - \Psi^{(-)} \frac{\partial_x^2 \Psi_{\lambda_1}^{(-)}}{\Psi_{\lambda_1}^{(-)}} = \Psi^{(-)} (\lambda_1 - \lambda).$$

therefore, differentiating $\Psi^{(+)}$, we have

$$\begin{aligned}\partial_x \Psi^{(+)} &= \partial_x^2 \Psi^{(-)} - \partial_x \Psi^{(-)} \frac{\partial_x \Psi_{\lambda_1}^{(-)}}{\Psi_{\lambda_1}^{(-)}} - \Psi^{(-)} \frac{\partial_x^2 \Psi_{\lambda_1}^{(-)}}{\Psi_{\lambda_1}^{(-)}} + \Psi^{(-)} \left(\frac{\partial_x \Psi_{\lambda_1}^{(-)}}{\Psi_{\lambda_1}^{(-)}} \right)^2 \\ &= \Psi^{(-)} (\lambda_1 - \lambda) + \Psi^{(-)} \left(\frac{\partial_x \Psi_{\lambda_1}^{(-)}}{\Psi_{\lambda_1}^{(-)}} \right)^2 - \partial_x \Psi^{(-)} \frac{\partial_x \Psi_{\lambda_1}^{(-)}}{\Psi_{\lambda_1}^{(-)}},\end{aligned}$$

By differentiating $\partial_x \Psi^{(+)}$, we have

$$\begin{aligned}\partial_x^2 \Psi^{(+)} &= \partial_x \Psi^{(-)} (\lambda_1 - \lambda) + 2 \partial_x \Psi^{(-)} \left(\frac{\partial_x \Psi_{\lambda_1}^{(-)}}{\Psi_{\lambda_1}^{(-)}} \right)^2 + 2 \Psi^{(-)} \frac{\partial_x \Psi_{\lambda_1}^{(-)}}{\Psi_{\lambda_1}^{(-)}} \frac{\partial_x^2 \Psi_{\lambda_1}^{(-)}}{\Psi_{\lambda_1}^{(-)}} \\ &\quad - 2 \Psi^{(-)} \left(\frac{\partial_x \Psi_{\lambda_1}^{(-)}}{\Psi_{\lambda_1}^{(-)}} \right)^3 - \partial_x^2 \Psi^{(-)} \frac{\partial_x \Psi_{\lambda_1}^{(-)}}{\Psi_{\lambda_1}^{(-)}} - \partial_x \Psi^{(-)} \frac{\partial_x^2 \Psi_{\lambda_1}^{(-)}}{\Psi_{\lambda_1}^{(-)}} \\ &= \partial_x \Psi^{(-)} (\lambda_1 - \lambda) + 2 \partial_x \Psi^{(-)} \left(\frac{\partial_x \Psi_{\lambda_1}^{(-)}}{\Psi_{\lambda_1}^{(-)}} \right)^2 - 2 \Psi^{(-)} \left(\frac{\partial_x \Psi_{\lambda_1}^{(-)}}{\Psi_{\lambda_1}^{(-)}} \right)^3 - \partial_x \Psi^{(-)} \frac{\partial_x^2 \Psi_{\lambda_1}^{(-)}}{\Psi_{\lambda_1}^{(-)}} \\ &\quad - \frac{\partial_x \Psi_{\lambda_1}^{(-)}}{\Psi_{\lambda_1}^{(-)}} \left(\partial_x^2 \Psi^{(-)} - \Psi^{(-)} \frac{\partial_x^2 \Psi_{\lambda_1}^{(-)}}{\Psi_{\lambda_1}^{(-)}} \right) + \Psi^{(-)} \frac{\partial_x \Psi_{\lambda_1}^{(-)}}{\Psi_{\lambda_1}^{(-)}} \frac{\partial_x^2 \Psi_{\lambda_1}^{(-)}}{\Psi_{\lambda_1}^{(-)}} \\ &= \partial_x \Psi^{(-)} (\lambda_1 - \lambda) + 2 \partial_x \Psi^{(-)} \left(\frac{\partial_x \Psi_{\lambda_1}^{(-)}}{\Psi_{\lambda_1}^{(-)}} \right)^2 - 2 \Psi^{(-)} \left(\frac{\partial_x \Psi_{\lambda_1}^{(-)}}{\Psi_{\lambda_1}^{(-)}} \right)^3 - \partial_x \Psi^{(-)} \frac{\partial_x^2 \Psi_{\lambda_1}^{(-)}}{\Psi_{\lambda_1}^{(-)}} \\ &\quad - \Psi^{(-)} \frac{\partial_x \Psi_{\lambda_1}^{(-)}}{\Psi_{\lambda_1}^{(-)}} (\lambda_1 - \lambda) + \Psi^{(-)} \frac{\partial_x \Psi_{\lambda_1}^{(-)}}{\Psi_{\lambda_1}^{(-)}} \frac{\partial_x^2 \Psi_{\lambda_1}^{(-)}}{\Psi_{\lambda_1}^{(-)}} \\ &= (\lambda_1 - \lambda) \left(\partial_x \Psi^{(-)} - \Psi^{(-)} \frac{\partial_x \Psi_{\lambda_1}^{(-)}}{\Psi_{\lambda_1}^{(-)}} \right) - \frac{\partial_x^2 \Psi_{\lambda_1}^{(-)}}{\Psi_{\lambda_1}^{(-)}} \left(\partial_x \Psi^{(-)} - \Psi^{(-)} \frac{\partial_x \Psi_{\lambda_1}^{(-)}}{\Psi_{\lambda_1}^{(-)}} \right) \\ &\quad + 2 \left(\frac{\partial_x \Psi_{\lambda_1}^{(-)}}{\Psi_{\lambda_1}^{(-)}} \right)^2 \left(\partial_x \Psi^{(-)} - \Psi^{(-)} \frac{\partial_x \Psi_{\lambda_1}^{(-)}}{\Psi_{\lambda_1}^{(-)}} \right) \\ &= \left(2 \left(\frac{\partial_x \Psi_{\lambda_1}^{(-)}}{\Psi_{\lambda_1}^{(-)}} \right)^2 - \frac{\partial_x^2 \Psi_{\lambda_1}^{(-)}}{\Psi_{\lambda_1}^{(-)}} + \lambda_1 - \lambda \right) \left(\partial_x \Psi^{(-)} - \Psi^{(-)} \frac{\partial_x \Psi_{\lambda_1}^{(-)}}{\Psi_{\lambda_1}^{(-)}} \right) \\ &= \left(2 \left(\frac{\partial_x \Psi_{\lambda_1}^{(-)}}{\Psi_{\lambda_1}^{(-)}} \right)^2 - \frac{\partial_x^2 \Psi_{\lambda_1}^{(-)}}{\Psi_{\lambda_1}^{(-)}} + \lambda_1 - \lambda \right) \Psi^{(+)} \\ &= \left(2 \left(\frac{\partial_x \Psi_{\lambda_1}^{(-)}}{\Psi_{\lambda_1}^{(-)}} \right)^2 - V_- + 2\lambda_1 - \lambda \right) \Psi^{(+)},\end{aligned}$$

which implies that, for $\lambda \neq \lambda_1$, the function $\Psi^{(+)}$ satisfies the Schrödinger equation $H_+ \Psi^{(+)} = \lambda \Psi^{(+)}$,

$$V_+ = 2 \left(\frac{\partial_x \Psi_{\lambda_1}^{(-)}}{\Psi_{\lambda_1}^{(-)}} \right)^2 - V_- + 2\lambda_1 = V_- - 2 \partial_x \left(\frac{\partial_x \Psi_{\lambda_1}^{(-)}}{\Psi_{\lambda_1}^{(-)}} \right) = \Psi_{\lambda_1}^{(-)} \partial_x^2 \left(\frac{1}{\Psi_{\lambda_1}^{(-)}} \right) + \lambda_1.$$

□

Remark 4.2. According to the corollary of the general Darboux transformation, see section 1, we have $m = -\lambda$, $m_1 = -\lambda_1$, $f(x) = V_-$, $y = \Psi^{(-)}$ and $u = y = \Psi^{(+)}$. This gives us a faster proof of theorem 4.1.

In agreement with the previous theorem, we obtain the following results.

Proposition 4.3. *DT is isogaloisian and virtually strong isogaloisian. Furthermore, if $\partial_x(\ln \Psi_{\lambda_1}^{(-)}) \in K$, then DT is strong isogaloisian.*

PROOF. Let K and L_λ be respectively the coefficient field and the Picard-Vessiot extension of the equation \mathcal{L}_λ . Similarly, let \tilde{K} , \tilde{L}_λ be the coefficient field and the Picard-Vessiot extension of the equation $\tilde{\mathcal{L}}_\lambda$. As $\text{DT}(V_-) = V_+ = 2W^2 - V_- - 2\lambda_1$, where $W = -\partial_x(\ln \Psi_{\lambda_1}^{(-)})$, we have $\tilde{K} = K \langle \partial_x(\ln \Psi_{\lambda_1}^{(-)}) \rangle$. The Riccati equation $\partial_x W = V_- - W^2$ has one algebraic solution (Singer 1981, [57]), in this case $W = -\partial_x(\ln \Psi_{\lambda_1}^{(-)})$. Let $\langle \Psi_{(1,\lambda)}^{(-)}, \Psi_{(2,\lambda)}^{(-)} \rangle$ be a basis of solutions for equation \mathcal{L}_λ and $\langle \Psi_{(1,\lambda)}^{(+)}, \Psi_{(2,\lambda)}^{(+)} \rangle$ a basis of solutions for equation $\tilde{\mathcal{L}}_\lambda$. Since the coefficient field for equation $\tilde{\mathcal{L}}_\lambda$ is $\tilde{K} = K \langle \partial_x(\ln \Psi_{\lambda_1}^{(-)}) \rangle$, we have $L = K \langle \Psi_{(1,\lambda)}^{(-)}, \Psi_{(2,\lambda)}^{(-)} \rangle$ and

$$\begin{aligned} \tilde{L} &= \tilde{K} \langle \Psi_{(1,\lambda)}^{(+)}, \Psi_{(2,\lambda)}^{(+)} \rangle = K \langle \Psi_{(1,\lambda)}^{(+)}, \Psi_{(2,\lambda)}^{(+)}, \partial_x(\ln \Psi_{\lambda_1}^{(-)}) \rangle \\ &= K \langle \Psi_{(1,\lambda)}^{(-)}, \Psi_{(2,\lambda)}^{(-)}, \partial_x(\ln \Psi_{\lambda_1}^{(-)}) \rangle = \tilde{K} \langle \Psi_{(1,\lambda)}^{(-)}, \Psi_{(2,\lambda)}^{(-)} \rangle, \end{aligned}$$

for $\lambda = \lambda_1$ and for $\lambda \neq \lambda_1$. Since $\partial_x(\ln \Psi_{\lambda_1}^{(-)})$ is algebraic over K , then

$$(\text{DGal}(L_\lambda/K))^0 = (\text{DGal}(\tilde{L}_\lambda/K))^0, \quad \text{DGal}(L_\lambda/K) = \text{DGal}(\tilde{L}_\lambda/\tilde{K}),$$

which means that DT is a virtually strong and isogaloisian transformation.

In the case $\partial_x(\ln \Psi_{\lambda_1}^{(-)}) \in K$, then $\tilde{K} = K$ and $\tilde{L} = L$, which means that DT is a strong isogaloisian transformation. □

Proposition 4.4. *Consider Schrödinger operators $\mathfrak{L}_\lambda := H_- - \lambda$ and $\tilde{\mathfrak{L}}_\lambda := H_+ - \lambda$ such that $\text{DT}(H_- - \lambda) = H_+ - \lambda$. The eigenrings of \mathfrak{L}_λ and $\tilde{\mathfrak{L}}_\lambda$ are isomorphic.*

PROOF. Let $\mathcal{E}(\mathfrak{L}_\lambda)$ and $\mathcal{E}(\tilde{\mathfrak{L}}_\lambda)$ denote the eigenrings of \mathfrak{L}_λ and $\tilde{\mathfrak{L}}_\lambda$ respectively. By proposition 4.3 the connected identity component of the Galois group is preserved by Darboux transformation. Let $T \in \mathcal{E}(\mathfrak{L}_\lambda)$. Let $\text{Sol}(\mathfrak{L}_\lambda)$ and $\text{Sol}(\tilde{\mathfrak{L}}_\lambda)$ be the solution spaces for $\mathfrak{L}_\lambda \Psi^{(-)} = 0$ and $\tilde{\mathfrak{L}}_\lambda \Psi^{(-)} = 0$ respectively. To transform $\mathcal{E}(\mathfrak{L}_\lambda)$ into $\mathcal{E}(\tilde{\mathfrak{L}}_\lambda)$, we follow the commuting diagram:

$$\begin{array}{ccc} \text{Sol}(\mathfrak{L}_\lambda) & \xrightarrow{T} & \text{Sol}(\mathfrak{L}_\lambda) \\ A^\dagger \uparrow & & \downarrow A \\ \text{Sol}(\tilde{\mathfrak{L}}_\lambda) & \xrightarrow{\tilde{T}} & \text{Sol}(\tilde{\mathfrak{L}}_\lambda) \end{array} \quad \text{so} \quad \tilde{T} = AT A^\dagger \text{mod}(\tilde{\mathfrak{L}}_\lambda) \in \mathcal{E}(\tilde{\mathfrak{L}}_\lambda),$$

where A^\dagger and A are the raising and lowering operators. □

Example. Consider the Schrödinger equation \mathcal{L}_λ with potential $V = V_- = 0$, which means that $\Lambda = \mathbb{C}$. If we choose $\lambda_1 = 0$ and as particular solution $\Psi_0^{(-)} = x$, then for $\lambda \neq 0$ the general solution is given by

$$\Psi_\lambda^{(-)} = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}.$$

Applying the Darboux transformation DT, we have $\text{DT}(\mathcal{L}_\lambda) = \tilde{\mathcal{L}}_\lambda$, where

$$\text{DT}(V_-) = V_+ = \frac{2}{x^2}$$

and for $\lambda \neq 0$

$$\text{DT}(\Psi_\lambda^{(-)}) = \Psi_\lambda^{(+)} = \frac{c_1(\sqrt{-\lambda}x - 1)e^{\sqrt{-\lambda}x}}{x} - \frac{c_2(\sqrt{-\lambda}x + 1)e^{-\sqrt{-\lambda}x}}{x}.$$

We can see that $\tilde{K} = K = \mathbb{C}(x)$ for all $\lambda \in \Lambda$ and the Picard-Vessiot extensions are given by $L_0 = \tilde{L}_0 = \mathbb{C}(x)$, $L_\lambda = \tilde{L}_\lambda = \mathbb{C}(x, e^{\sqrt{\lambda}x})$ for $\lambda \in \mathbb{C}^*$. It follows that $\text{DGal}(L_0/K) = \text{DGal}(\tilde{L}_0/K) = e$; for $\lambda \neq 0$, we have $\text{DGal}(L_\lambda/K) = \text{DGal}(\tilde{L}_\lambda/K) = \mathbb{G}_m$. The eigenrings of the operators \mathfrak{L}_0 and \mathfrak{L}_λ are given by

$$\mathcal{E}(\mathfrak{L}_0) = \text{Vect}(1, \partial_x, x\partial_x - 1, x^2\partial_x - x),$$

$$\mathcal{E}(\tilde{\mathfrak{L}}_0) = \text{Vect}\left(1, x\partial_x - 1, x^4\partial_x - 2x^3, \frac{\partial_x}{x^2} + \frac{1}{x^3}\right)$$

and for $\lambda \neq 0$

$$\mathcal{E}(\mathfrak{L}_\lambda) = \text{Vect}(1, \partial_x), \quad \mathcal{E}(\tilde{\mathfrak{L}}_\lambda) = \text{Vect}\left(1, -\left(\lambda + \frac{1}{x^2}\right)\partial_x - \frac{1}{x^3}\right),$$

where $\mathcal{L}_\lambda := \mathfrak{L}_\lambda \Psi^{(-)} = 0$ and $\tilde{\mathcal{L}}_\lambda := \tilde{\mathfrak{L}}_\lambda \Psi^{(+)} = 0$.

Applying iteratively the Darboux transformation, theorem 4.1, and by propositions 4.3, 4.4 we have the following results.

Proposition 4.5 (Galoisian version of DT_n). *We continue with the notations of theorem 4.1. We set $V_0 = V_-$, $H^{(0)} = H_-$, $\Psi^{(0)} = \Psi^{(-)}$. We still assume that the algebraic spectrum Λ is non-empty. We consider the operators $\mathcal{L}_\lambda^{(n)}$ (given by $H^{(n)}\Psi^{(n)} = \lambda\Psi^{(n)}$) obtained by successive Darboux transformations. We let K_n be the coefficient field of $\mathcal{L}_\lambda^{(n)}$ (i.e. $V_n \in K_n$) and $K = K_0$. Let $L_\lambda^{(n)}$ denote the Picard-Vessiot extension of $\mathcal{L}_\lambda^{(n)}$. The Darboux transformation DT_n is such that $\mathcal{L}_\lambda^{(n)} \mapsto \mathcal{L}_\lambda^{(n+1)}$, $V_n \mapsto V_{n+1}$, $\Psi_\lambda^{(n)} \mapsto \Psi_\lambda^{(n+1)}$ and $L_\lambda^{(n+1)}$ the Picard-Vessiot extension of $\mathcal{L}_\lambda^{(n+1)}$.*

Then the following statements hold:

- i): $\text{DT}_n(V_-) = \text{DT}(V_n) = V_{n+1} = V_n - 2\partial_x^2 \left(\ln \Psi_\lambda^{(n)} \right) = V_- - 2 \sum_{k=0}^n \partial_x^2 \left(\ln \Psi_{\lambda_k}^{(k)} \right)$,
 where $\Psi_{\lambda_k}^{(k)}$ is a particular solution for $\lambda = \lambda_k$, $k = 0, \dots, n$. In particular, if $\lambda_n = \lambda_0$ and $\Lambda = \mathbb{C}$, then there exists $\Psi_{\lambda_n}^{(n)}$ such that $V_n \neq V_{n-2}$, with $n \geq 2$.
- ii): $\text{DT}(\Psi_\lambda^{(n)}) = \text{DT}_n(\Psi_\lambda^{(-)}) = \Psi_\lambda^{(n+1)} = \partial_x \Psi_\lambda^{(n)} - \Psi_\lambda^{(n)} \frac{\partial_x \Psi_{\lambda_n}^{(n)}}{\Psi_{\lambda_n}^{(n)}} = \frac{W(\Psi_{\lambda_n}^{(n)}, \Psi_\lambda^{(n)})}{W(\Psi_{\lambda_n}^{(n)})}$
 where $\Psi_\lambda^{(n)}$ is a general solution for $\lambda \in \Lambda \setminus \{\lambda_n\}$ of $\mathcal{L}_\lambda^{(n)}$.

- iii): $K_{n+1} = K_n \left\langle \partial_x (\ln \Psi_{\lambda_n}^{(n)}) \right\rangle$.
- iv): DT_n is isogaloisian and virtually strongly isogaloisian. Furthermore, if $\partial_x (\ln \Psi_{\lambda_n}^{(n)}) \in K_n$ then DT_n is strongly isogaloisian.
- v): The eigenrings of $H^{(n)} - \lambda$ and $H^{(n+1)} - \lambda$ are isomorphic.

PROOF. By induction on theorem 4.1 we obtain i) and ii). By induction on proposition 4.3 we obtain iii) and iv). By induction on proposition 4.4 we obtain v). \square

Example. Starting with $V = 0$, the following potentials can be obtained using Darboux iteration DT_n (see [10, 12]).

$$\begin{aligned} I) \ V_n &= \frac{n(n-1)b^2}{(bx+c)^2}, & II) \ V_n &= \frac{m^2n(n-1)(b^2-a^2)}{(a \cosh(mx) + b \sinh(mx))^2}, \\ III) \ V_n &= \frac{-4abm^2n(n-1)}{(ae^{mx} + be^{-mx})^2}, & IV) \ V_n &= \frac{m^2n(n-1)(b^2+a^2)}{(a \cos(mx) + b \sin(mx))^2}. \end{aligned}$$

In particular for the rational potential given in I), we have $K = K_n = \mathbb{C}(x)$ and for $\lambda_n = \lambda = 0$, we have

$$\Psi_0^{(n)} = \frac{c_1}{(bx+c)^n} + c_2(bx+c)^{n+1}, \text{ so that } \text{DGal}(L_0/K) = \text{DGal}(L_0^{(n)}/K) = e,$$

$$\mathcal{E}(H^{(n)}) = \text{Vect} \left(1, x\partial_x - 1, x^{2n+2}\partial_x - (n+1)x^{2n+1}, \frac{\partial_x}{x^{2n}} + \frac{n}{x^{2n+1}} \right),$$

whilst for $\lambda \neq 0$ and $\lambda_n = 0$, the general solution $\Psi_\lambda^{(n)}$ is given by

$$\Psi_\lambda^{(n)}(x) = c_1 f_n(x, \lambda) h_n(\sin(\sqrt{\lambda}x) + c_2 g_n(x, \lambda) j_n(\cos(\sqrt{\lambda}x)),$$

where $f_n, g_n, h_n, j_n \in \mathbb{C}(x)$, so that

$$\text{DGal}(L_\lambda/K) = \text{DGal}(L_\lambda^{(n)}/K) = \mathbb{G}_m,$$

and

$$\dim_{\mathbb{C}} \mathcal{E}(H - \lambda) = \dim_{\mathbb{C}} \mathcal{E}(H^{(n)} - \lambda) = 2.$$

4.2. Crum Iteration.

Proposition 4.6 (Galoisian version of CI_n). *Consider \mathcal{L}_λ given by $H\Psi = \lambda\Psi$, $H = -\partial_x^2 + V$, $V \in K$, such that $\text{Card}(\Lambda) > n$ for a fixed $n \in \mathbb{Z}_+$. Let $\mathcal{L}_\lambda^{(n)}$ be given by $H^{(n)}\Psi^{(n)} = \lambda\Psi^{(n)}$, where $H^{(n)} = \partial_x^2 + V_n$, $V_n \in K_n$. Let CI_n be the transformation such that $\mathcal{L}_\lambda \mapsto \mathcal{L}_\lambda^{(n)}$, $V \mapsto V_n$, $(\Psi_{\lambda_1}, \dots, \Psi_{\lambda_n}, \Psi_\lambda) \mapsto \Psi_\lambda^{(n)}$, where for $k = 1, \dots, n$ and the equation \mathcal{L}_λ , the function Ψ_λ is the general solution for $\lambda \neq \lambda_k$ and Ψ_{λ_k} is a particular solution for $\lambda = \lambda_k$. Then the following statements holds:*

- i): $\text{CI}_n(\mathcal{L}_\lambda) = \mathcal{L}_\lambda^{(n)}$ where $\text{CI}_n(V) = V_n = V - 2\partial_x^2 (\ln W(\Psi_{\lambda_1}, \dots, \Psi_{\lambda_n}))$ and

$$\text{CI}_n(\Psi_\lambda) = \Psi_\lambda^{(n)} = \frac{W(\Psi_{\lambda_1}, \dots, \Psi_{\lambda_n}, \Psi_\lambda)}{W(\Psi_{\lambda_1}, \dots, \Psi_{\lambda_n})},$$

where $\Psi_\lambda^{(n)}$ is the general solution of $\mathcal{L}_\lambda^{(n)}$.

- ii): $K_n = K \langle \partial_x (\ln W(\Psi_{\lambda_1}, \dots, \Psi_{\lambda_n})) \rangle$.

iii): CI_n is isogaloisian and virtually strongly isogaloisian. Furthermore, if

$$\partial_x (\ln W(\Psi_{\lambda_1}, \dots, \Psi_{\lambda_n})) \in K_n,$$

then CI_n is strongly isogaloisian.

iv): The eigenrings of $H - \lambda$ and $H^{(n)} - \lambda$ are isomorphic.

PROOF. By induction on theorem 4.1 we obtain i). By induction on proposition 4.3 we obtain ii) and iii). By induction on proposition 4.4 we obtain iv). \square

Example. To illustrate the Crum iteration with rational potentials, we consider $V = \frac{2}{x^2}$. The general solution of $\mathcal{L}_\lambda := H\Psi = \lambda\Psi$ is

$$\frac{c_1 e^{kx}(kx-1)}{x} + \frac{c_2 e^{-kx}(kx+1)}{x}, \quad \lambda = -k^2,$$

the eigenfunctions for $\lambda_1 = -1$, and $\lambda_2 = -4$, are respectively given by

$$\Psi_{-1} = \frac{e^{-x}(x+1)}{x}, \quad \Psi_{-4} = \frac{e^{-2x}(2x+1)}{2x}.$$

Thus, we obtain

$$\text{CI}_2(V) = V_2 = \frac{8}{(2x+3)^2}$$

and the general solution of $\mathcal{L}_\lambda^{(2)} := H^{(2)}\Psi^{(2)} = \lambda\Psi^{(2)}$ is

$$\text{CI}_2(\Psi_\lambda) = \Psi_\lambda^{(2)} = \frac{c_1 (k(2x+3)-2) e^{kx}}{2x+3} + \frac{c_2 (2+k(2x+3)) e^{-kx}}{4x+6}, \quad \lambda = -k^2.$$

The differential Galois groups and eigenrings are given by:

$$\text{DGal}(L_0/K) = \text{DGal}(L_0^{(2)}/K) = e, \quad \dim_{\mathbb{C}} \mathcal{E}(H) = \dim_{\mathbb{C}} \mathcal{E}(H^{(2)}) = 4,$$

and for $\lambda \neq 0$

$$\text{DGal}(L_\lambda/K) = \text{DGal}(L_\lambda^{(2)}/K) = \mathbb{G}_m, \quad \dim_{\mathbb{C}} \mathcal{E}(H - \lambda) = \dim_{\mathbb{C}} \mathcal{E}(H^{(2)} - \lambda) = 2.$$

Proposition 4.7. *The supersymmetric partner potentials V_\pm are rational functions if and only if the superpotential W is a rational function.*

PROOF. The supersymmetric partner potentials V_\pm are written as $V_\pm = W^2 \pm \partial_x W$. We start considering the superpotential $W \in \mathbb{C}(x)$, so trivially we have $V_\pm \in \mathbb{C}(x)$. Now assuming that $V_\pm \in \mathbb{C}(x)$ we have $\partial_x W \in \mathbb{C}(x)$ and $W^2 \in \mathbb{C}(x)$, which implies that $W\partial_x W \in \mathbb{C}(x)$ and therefore $W \in \mathbb{C}(x)$. \square

Corollary 4.8. *The superpotential $W \in \mathbb{C}(x)$ if and only if DT is strong isogaloisian.*

PROOF. Assume that the superpotential $W \in \mathbb{C}(x)$. Thus, by proposition 4.3, DT is strong isogaloisian. Now, assume that DT is strong isogaloisian. Thus, $V_\pm \in \mathbb{C}(x)$ and by proposition 4.7 we have $W \in \mathbb{C}(x)$. \square

4.3. Shape Invariance. The following definition is a partial Galoisian adaptation of the original definition given in [28] ($K = \mathbb{C}(x)$). The complete Galoisian adaptation is given when K is any differential field.

Definition 4.9 (Rational Shape Invariant Potentials). Assume $V_{\pm}(x; \mu) \in \mathbb{C}(x; \mu)$, where μ is a family of parameters. The potential $V = V_- \in \mathbb{C}(x)$ is called a rational shape invariant potential with respect to μ and $E = E_n$ if there exists f such that

$$V_+(x; a_0) = V_-(x; a_1) + R(a_1), \quad a_1 = f(a_0), \quad E_n = \sum_{k=2}^{n+1} R(a_k), \quad E_0 = 0.$$

Remark 4.10. We propose the following steps to check whether $V \in \mathbb{C}(x)$ is shape invariant.

Step 1.: Introduce parameters in $W(x)$ to obtain $W(x; \mu)$, write $V_{\pm}(x; \mu) = W^2(x; \mu) \pm \partial_x W(x; \mu)$, and replace μ by a_0 and a_1 .

Step 2.: Obtain polynomials $\mathcal{P} \in \mathbb{C}[x; a_0, a_1]$ and $\mathcal{Q} \in \mathbb{C}[x; a_0, a_1]$ such that

$$\partial_x(V_+(x; a_0) - V_-(x; a_1)) = \frac{\mathcal{P}(x; a_0, a_1)}{\mathcal{Q}(x; a_0, a_1)}.$$

Step 3.: Set $\mathcal{P}(x; a_0, a_1) \equiv 0$, as polynomial in x , to obtain a_1 in function of a_0 , i.e., $a_1 = f(a_0)$. Also obtain $R(a_1) = V_+(x; a_0) - V_-(x; a_1)$ and verify that exists $k \in \mathbb{Z}^+$ such that $R(a_1) + \dots + R(a_k) \neq 0$.

Example. Consider the superpotential of the three dimensional harmonic oscillator $W(r; \ell) = r - \frac{\ell+1}{r}$. By step 1, the supersymmetric partner potentials are

$$V_-(r; \ell) = r^2 + \frac{\ell(\ell+1)}{r^2} - 2\ell - 3, \quad V_+(r; \ell) = r^2 + \frac{(\ell+1)(\ell+2)}{r^2} - 2\ell - 1.$$

By step 2, we have $\partial_r(V_+(r; a_0) - V_-(r; a_1)) = -2\frac{a_0^2+3a_0-a_1^2-a_1+2}{r^3}$. By step 3, $(a_0+1)(a_0+2) = a_1(a_1+1)$, so that $a_1 = f(a_0) = a_0+1$, $a_n = f(a_{n-1}) = a_0+n$, $R(a_1) = 2$. Thus, we obtain the energy levels $E_n = 2n$ and the wave functions $\Psi_n^{(-)}(r; \ell) = A^\dagger(r; \ell) \dots A^\dagger(r; \ell+n-1)\Psi_0^{(-)}(r; \ell+n)$, compare with [23].

By theorem 4.1 and propositions 4.3, 4.4 and 4.7 we have the following result.

Theorem 4.11. Consider $\mathcal{L}_n := H\Psi^{(-)} = E_n\Psi^{(-)}$ with Picard-Vessiot extension L_n , where $n \in \mathbb{Z}_+$. If $V = V_- \in \mathbb{C}(x)$ is a shape invariant potential with respect to $E = E_n$, then

$$\text{DGal}(L_{n+1}/K) = \text{DGal}(L_n/K), \quad \mathcal{E}(H - E_{n+1}) \simeq \mathcal{E}(H - E_n), \quad n > 0.$$

Remark 4.12. As $W \in \mathbb{C}(x)$, a differential automorphism σ commutes with the raising and lowering operators A and A^\dagger . Furthermore the wave functions $\Psi_n^{(-)}$ can be written as $\Psi_n^{(-)} = P_n f_n \Psi_0^{(-)}$, where P_n is a polynomial of degree n in x and f_n is a sequence of functions with $f_0(x) = 1$ such as was shown in the case of Harmonic oscillators and Coulomb potentials.

5. Schrödinger Equations with Rational Potentials

Along this section we take $K = \mathbb{C}(x)$ to be our coefficient field. So all our differential operators will have coefficients in $K = \mathbb{C}(x)$.

5.1. Polynomial Potentials. We start by considering the Schrödinger equation (3.1) with polynomial potentials, i.e. $V \in \mathbb{C}[x]$, see [13, 64]. For simplicity and without loss of generality, we consider monic polynomials (if the polynomial is $c_n x^n + \dots + c_1 x + c_0$, apply the transformation $x \mapsto \sqrt[n+2]{c_n} x$).

When V is a polynomial of odd degree, is well known that the differential Galois group of such a Schrödinger equation (3.1) is $\mathrm{SL}(2, \mathbb{C})$, see [36].

We present here the complete result for the Schrödinger equation (3.1) with non-constant polynomial potential (Theorem 5.2), see also [4, §2]. The following lemma is useful for our purposes.

Lemma 5.1 (Completing Squares, [4]). *Every even degree monic polynomial of can be written in one only way completing squares, that is,*

$$(5.1) \quad Q_{2n}(x) = x^{2n} + \sum_{k=0}^{2n-1} q_k x^k = \left(x^n + \sum_{k=0}^{n-1} a_k x^k \right)^2 + \sum_{k=0}^{n-1} b_k x^k,$$

where

$$\begin{aligned} a_{n-1} &= \frac{q_{2n-1}}{2}, \quad a_{n-2} = \frac{q_{2n-2} - a_{n-1}^2}{2}, \quad a_{n-3} = \frac{q_{2n-3} - 2a_{n-1}a_{n-2}}{2}, \dots, \\ a_0 &= \frac{q_n - 2a_1a_{n-1} - 2a_2a_{n-2} - \dots}{2}, \quad b_0 = q_0 - a_0^2, \quad b_1 = q_1 - 2a_0a_1, \quad \dots, \\ b_{n-1} &= q_{n-1} - 2a_0a_{n-1} - 2a_1a_{n-2} - \dots. \end{aligned}$$

PROOF. See lemma 2.4 in [4, p. 275]. □

We remark that $V(x)$ as in equation (5.1) can be written in terms of the superpotential $W(x)$, i.e., $V(x) = W^2(x) - \partial_x W(x)$, when

$$nx^{n-1} + \sum_{k=1}^{n-1} k a_k x^{k-1} = - \sum_{k=0}^{n-1} b_k x^k$$

and $W(x)$ is given by

$$x^n + \sum_{k=0}^{n-1} a_k x^k.$$

The following theorem also can be found in [4, §2], see also [3]. Here we present a quantum mechanics adapted version.

Theorem 5.2 (Polynomial potentials and Galois groups, [4]). *Let us consider the Schrödinger equation (3.1), with $V(x) \in \mathbb{C}[x]$ a polynomial of degree $k > 0$. Then, its differential Galois group $\mathrm{DGal}(L_\lambda/K)$ falls in one of the following cases:*

- (1) $\text{DGal}(L_\lambda/K) = \text{SL}(2, \mathbb{C})$,
- (2) $\text{DGal}(L_\lambda/K) = \mathbb{B}$,

and the eigenring of $H - \lambda$ is trivial, i.e., $\mathcal{E}(H - \lambda) = \text{Vect}(1)$. Furthermore, $\text{DGal}(L_\lambda/K) = \mathbb{B}$ if and only if the following conditions hold:

- (1) $V(x) - \lambda$ is a polynomial of degree $k = 2n$ writing in the form of equation (5.1).
- (2) $b_{n-1} - n$ or $-b_{n-1} - n$ is a positive even number $2m$, $m \in \mathbb{Z}_+$.
- (3) There exist a monic polynomial P_m of degree m , satisfying

$$\partial_x^2 P_m + 2 \left(x^n + \sum_{k=0}^{n-1} a_k x^k \right) \partial_x P_m + \left(n x^{n-1} + \sum_{k=0}^{n-2} (k+1) a_{k+1} x^k - \sum_{k=0}^{n-1} b_k x^k \right) P_m = 0,$$

or

$$\partial_x^2 P_m - 2 \left(x^n + \sum_{k=0}^{n-1} a_k x^k \right) \partial_x P_m - \left(n x^{n-1} + \sum_{k=0}^{n-2} (k+1) a_{k+1} x^k + \sum_{k=0}^{n-1} b_k x^k \right) P_m = 0.$$

In such cases, the only possibilities for eigenfunctions with rational superpotentials are given by

$$\Psi_\lambda = P_m e^{f(x)}, \quad \text{or} \quad \Psi_\lambda = P_m e^{-f(x)}, \quad \text{where } f(x) = \frac{x^{n+1}}{n+1} + \sum_{k=0}^{n-1} \frac{a_k x^{k+1}}{k+1}.$$

PROOF. See theorem 2.5 in [4, p. 276]. □

An easy consequence of the above theorem is the following.

Corollary 5.3. *Assume that $V(x)$ is an algebraically solvable polynomial potential. Then $V(x)$ is of degree 2.*

PROOF. Writing $V(x) - \lambda$ in the form of equation (5.1) we see that $b_{n-1} - n = 2m$ or $-b_{n-1} - n = 2m$, where $m \in \mathbb{Z}_+$. Thus, the integrability of the Schrödinger equation with $\text{Card}(\Lambda) > 1$ is obtained when b_{n-1} is constant, so $n = 1$. □

Remark 5.4. Given a polynomial potential $V(x)$ such that $\text{Spec}_p(H) \cap \Lambda \neq \emptyset$, we can obtain bound states and normalized wave functions if and only if the potential $V(x)$ is a polynomial of degree $4n + 2$. Furthermore, one integrability condition of $H\Psi = \lambda\Psi$ for $\lambda \in \Lambda$ is that b_{2n} must be an odd integer. In particular, if the potential

$$V(x) = x^{4n+2n} + \mu x^{2n}, \quad n > 0$$

is a quasi-exactly solvable, then μ is an odd integer. For this kind of potentials, we obtain bound states only when μ is a negative odd integer.

On another hand, the non-constant polynomial potentials $V(x)$ of degree $4n$ are associated to non-hermitian Hamiltonians and \mathcal{PT} invariance which are not considered here, see [7]. Furthermore, one integrability condition of $H\Psi = \lambda\Psi$ for

$\lambda \in \Lambda$ is that b_{2n-1} must be an even integer. In particular, if the Schrödinger equation

$$H\Psi = \lambda\Psi, \quad V(x) = x^{4n} + \mu x^{2n-1}, \quad \lambda \in \Lambda$$

is integrable, then μ is an even integer.

We present the following examples to illustrate the previous theorem and remark.

Weber's Equation and Harmonic Oscillator. The Schrödinger equation with potential $V(x) = x^2 + q_1x + q_0$ corresponds to the Rehm form of the Weber's equation. By lemma 5.1 we have

$$V(x) - \lambda = (x + a_0)^2 + b_0, \quad a_0 = q_1/2, \quad b_0 = q_0 - q_1^2/4 - \lambda.$$

So that we obtain $\pm b_0 - 1 = 2m$, where $m \in \mathbb{Z}_+$. If b_0 is an odd integer, then

$$\text{DGal}(L_\lambda/K) = \mathbb{B}, \quad \mathcal{E}(H - \lambda) = \text{Vect}(1), \quad \lambda \in \Lambda = \{\pm(2m+1) + q_0 - q_1^2/4 : m \in \mathbb{Z}_+\}$$

and the set of eigenfunctions is either

$$\Psi_\lambda = P_m e^{\frac{1}{2}(x^2 + q_1x)}, \quad \text{or}, \quad \Psi_\lambda = P_m e^{-\frac{1}{2}(x^2 + q_1x)}.$$

In the second case we have bound state wave function and $\text{Spec}_p(H) \cap \Lambda = \text{Spec}_p(H) = \{E_m = 2m + 1 + q_0 - q_1^2/4 : m \in \mathbb{Z}_+\}$, which is infinite. The polynomials P_m are related with the Hermite polynomials H_m , [15, 31, 46].

In particular we have the harmonic oscillator potential

$$V(x) = \frac{1}{4}\omega^2 x^2 - \frac{\omega}{2},$$

and the Schrödinger equation $H\Psi = E\Psi$. Through the change of independent variable $x \mapsto \sqrt{\frac{2}{\omega}}x$ we obtain $V(x) = x^2 - 1$ and $\lambda = \frac{2}{\omega}E$, that is, $q_1 = 0$ and $q_0 = -1$. It follows that $\Lambda = \{\pm(2m+1) - 1 : m \in \mathbb{Z}_+\}$ and the set of eigenfunctions is either

$$\Psi_\lambda = P_m e^{\frac{1}{2}x^2}, \quad \text{or}, \quad \Psi_\lambda = P_m e^{-\frac{1}{2}x^2},$$

where as below, $\text{DGal}(L_\lambda/K) = \mathbb{B}$ and $\mathcal{E}(H - \lambda) = \{1\}$ for all $\lambda \in \Lambda$. In the second case we have bound state wave function, $\text{Spec}_p(H) \cap \Lambda = \text{Spec}_p(H) = \Lambda_+ = \{2m : m \in \mathbb{Z}_+\}$ and $P_m = H_m$. The wave functions of $H\Psi = E\Psi$ for the harmonic oscillator potential are given by

$$\Psi_m = H_m \left(\sqrt{\frac{2}{\omega}}x \right) \Psi_0, \quad \Psi_0 = e^{-\frac{\omega}{4}x^2}, \quad E = E_m = m\omega.$$

Quartic and Sextic Anharmonic Oscillator. The Schrödinger equation with potential $V(x) = x^4 + q_3x^3 + q_2x^2 + q_1x + q_0$ can be obtained through transformations of confluent Heun's equation, though we will not adress this here. By lemma 5.1 we have

$$V(x) - \lambda = (x^2 + a_1x + a_0)^2 + b_1x + b_0,$$

where $a_1 = q_3/2$, $a_0 = q_2/2 - a_1^2/2$, $b_1 = q_1 - 2a_0a_1$ and $b_0 = q_0 - a_0^2 - \lambda$. So we obtain $\pm b_1 - 2 = 2m$, where $m \in \mathbb{Z}_+$. If $\Lambda \neq \emptyset$, then b_1 is an even integer, P_m satisfy

the relation (A.1) and $\text{DGal}(L_\lambda/K) = \mathbb{B}$ for all $\lambda \in \Lambda$. The set of eigenfunctions is either

$$\Psi_\lambda = P_m e^{\frac{x^3}{3} + \frac{a_1 x^2}{2} + a_0 x}, \text{ or, } \Psi_\lambda = P_m e^{-\left(\frac{x^3}{3} + \frac{a_1 x^2}{2} + a_0 x\right)},$$

where λ and m are related, which means that Λ is finite, i.e., the potential is algebraically quasi-solvable. In particular for $q_3 = 2\iota l$, $q_2 = l^2 - 2k$, $q_1 = 2\iota(lk - J)$ and $q_0 = 0$, we have the quartic anharmonic oscillator potential, which can be found in [7].

Now, considering the potentials $V(x, \mu) = x^4 + 4x^3 + 2x^2 - \mu x$, again by lemma 5.1 we have

$$V(x, \mu) - \lambda = (x^2 + 2x - 1)^2 + (4 - \mu)x - 1 - \lambda,$$

so that $\pm(4 - \mu) - 2 = 2n$, where $n \in \mathbb{Z}_+$ and in consequence $\mu \in 2\mathbb{Z}$. Such μ can be either $\mu = 2 - 2n$ or $\mu = 2n + 6$, where $n \in \mathbb{Z}_+$. By theorem 5.2, there exists a monic polynomial P_n satisfying respectively

$$\begin{aligned} \partial_x^2 P_n + (2x^2 + 4x - 2)\partial_x P_n + ((\mu - 2)x + 3 + \lambda)P_n &= 0, \quad \mu = 2 - 2n, \quad \text{or} \\ \partial_x^2 P_n - (2x^2 + 4x - 2)\partial_x P_n + ((\mu - 6)x - 1 + \lambda)P_n &= 0, \quad \mu = 2n + 6 \end{aligned}$$

for $\Lambda \neq \emptyset$. This algebraic relation between the coefficients of P_n , μ and λ give us the set Λ in the following way:

- (1) Write $P_n = x^n + c_{n-1}x^{n-1} + \dots + c_0$, where c_i are unknown.
- (2) Pick μ and replace P_n in the algebraic relation (A.1) to obtain a polynomial of degree n with $n+1$ undetermined coefficients involving c_0, \dots, c_{n-1} and λ . Each of such coefficients must be zero.
- (3) The term $n+1$ is linear in λ and c_{n-1} , thus we write c_{n-1} in terms of λ . After of the elimination of the term $n+1$, we replace c_{n-1} in the term n to obtain a quadratic polynomial in λ and so on until we reach the constant term which is a polynomial $Q_{n+1}(\lambda)$ of degree $n+1$ in λ . It follows that $\Lambda = \{\lambda : Q_{n+1}(\lambda) = 0\}$ and c_0, \dots, c_{n-1} are determined for each value of λ .

For $\mu = 2n + 6$, we have:

$$\begin{aligned} n = 0, \quad V(x, 6), \quad P_0 &= 1, \quad \Lambda = \{1\} \\ n = 1, \quad V(x, 8), \quad P_1 &= x + 1 \mp \sqrt{2}, \quad \Lambda = \{3 \pm 2\sqrt{2}\} \\ &\vdots \end{aligned}$$

and the set of eigenfunctions is

$$\Psi_{\lambda, \mu} = P_n e^{-\frac{1}{3}x^3 - x^2 + x}.$$

In the same way, we can obtain Λ , P_n and $\Psi_{\lambda, \mu}$ for $\mu = 2 - 2n$. However, we do not have bound states, $\text{Spec}_p(H) \cap \Lambda = \emptyset$, $\text{DGal}(L_\lambda/K) = \mathbb{B}$ and $\mathcal{E}(H - \lambda) = \text{Vect}(1)$ for all $\lambda \in \Lambda$.

The well known *sextic anharmonic oscillator* $x^6 + q_5x^5 + \dots + q_1x + q_0$ can be treated in a similar way, obtaining bound states wave functions and the Bender-Dunne orthogonal polynomials, which corresponds to $Q_{n+1}(\lambda)$, i.e. we also recover the results of [8, 29, 53]. The Schrödinger equation with this potential, under suitable transformations, also falls in the class of confluent Heun equations.

5.2. Rational Potentials and Kovacic's Algorithm. Here we apply Kovacic's algorithm (see appendix A) to solve the Schrödinger equation with rational (non-polynomial) potentials.

Three dimensional harmonic oscillator potential:

$$V(r) = \frac{1}{4}\omega^2 r^2 + \frac{\ell(\ell+1)}{r^2} - \left(\ell + \frac{3}{2}\right)\omega, \quad \ell \in \mathbb{Z}.$$

In this case, the Schrödinger equation is written as

$$\partial_r^2 \Psi = \left(\left(\frac{1}{2}\omega r \right)^2 + \frac{\ell(\ell+1)}{r^2} - \left(\ell + \frac{3}{2} \right) \omega - E \right) \Psi.$$

By the change $r \mapsto \left(\sqrt{\frac{2}{\omega}}\right) r$ we obtain the Schrödinger equation

$$\partial_r^2 \Psi = \left(r^2 + \frac{\ell(\ell+1)}{r^2} - (2\ell+3) - \lambda \right) \Psi, \quad \lambda = \frac{2}{\omega} E$$

and, in order to apply Kovacic's algorithm, we set

$$R = r^2 + \frac{\ell(\ell+1)}{r^2} - (2\ell+3) - \lambda.$$

We can see that this equation could fall in case 1, in case 2 or in case 4 of Kovacic's algorithm. We start discarding the case 2 because by step 1 (of Kovacic's algorithm) we should have conditions c_2 and ∞_3 ; so we should have $E_c = \{2, 4 + 4\ell, -4\ell\}$ and $E_\infty = \{-2\}$, and by step 2, we would have $n = -4 \notin \mathbb{Z}_+$, so that $D = \emptyset$. So this Schrödinger equation never falls in case 2. Now, we only work with case 1; by step 1, conditions c_2 and ∞_3 are satisfied, so that

$$\left[\sqrt{R} \right]_c = 0, \quad \alpha_c^\pm = \frac{1 \pm (2\ell+1)}{2}, \quad \left[\sqrt{R} \right]_\infty = r, \quad \alpha_\infty^\pm = \frac{\mp(\lambda + 2\ell + 3) - 1}{2}.$$

By step 2 we have the following possibilities for $n \in \mathbb{Z}_+$ and for $\lambda \in \Lambda$:

$$\Lambda_{++}) \quad n = \alpha_\infty^+ - \alpha_0^+ = -\frac{1}{2}(4\ell + 6 + \lambda), \quad \lambda = -2n - 4\ell - 6,$$

$$\Lambda_{+-}) \quad n = \alpha_\infty^+ - \alpha_0^- = -\frac{1}{2}(4 + \lambda), \quad \lambda = -2n - 4,$$

$$\Lambda_{-+}) \quad n = \alpha_\infty^- - \alpha_0^+ = \frac{\lambda}{2}, \quad \lambda = 2n,$$

$$\Lambda_{--}) \quad n = \alpha_\infty^- - \alpha_0^- = \frac{1}{2}(4\ell + 2 + \lambda), \quad \lambda = 2n - 4\ell - 2,$$

where $\Lambda_{++} \cup \Lambda_{+-} \cup \Lambda_{-+} \cup \Lambda_{--} = \Lambda$, which means that $\lambda = 2m$, $m \in \mathbb{Z}$. Now, for $\lambda \in \Lambda$, the rational function ω in Kovacic's algorithm is given by:

$$\Lambda_{++}) \quad \omega = r + \frac{\ell+1}{r}, \quad R_n = r^2 + \frac{\ell(\ell+1)}{r^2} + (2\ell+3) + 2n,$$

$$\Lambda_{+-}) \quad \omega = r - \frac{\ell}{r}, \quad R_n = r^2 + \frac{\ell(\ell+1)}{r^2} - (2\ell-1) + 2n,$$

$$\Lambda_{-+}) \quad \omega = -r + \frac{\ell+1}{r}, \quad R_n = r^2 + \frac{\ell(\ell+1)}{r^2} - (2\ell+3) - 2n,$$

$$\Lambda_{--}) \quad \omega = -r - \frac{\ell}{r}, \quad R_n = r^2 + \frac{\ell(\ell+1)}{r^2} + (2\ell-1) - 2n,$$

where R_n is the coefficient of the differential equation $\tilde{\mathcal{L}}_n := \partial_r^2 \Psi = R_n \Psi$, which is integrable for every n . For every $\lambda \in \Lambda$, we can see that $\text{DGal}(\tilde{L}_n/K) = \text{DGal}(L_\lambda/K)$, where $\mathcal{L}_\lambda := H\Psi = \lambda\Psi$.

By step 3, there exists a polynomial of degree n satisfying the relation (A.1):

$$\begin{aligned} \Lambda_{++}) \quad & \partial_r^2 P_n + 2 \left(r + \frac{\ell+1}{r} \right) \partial_r P_n - 2nP_n = 0, \quad \lambda \in \Lambda_-, \\ \Lambda_{+-}) \quad & \partial_r^2 P_n + 2 \left(r - \frac{\ell}{r} \right) \partial_r P_n - 2nP_n = 0, \quad \lambda \in \Lambda_-, \\ \Lambda_{-+}) \quad & \partial_r^2 P_n + 2 \left(-r + \frac{\ell+1}{r} \right) \partial_r P_n + 2nP_n = 0, \quad \lambda \in \Lambda_+, \\ \Lambda_{--}) \quad & \partial_r^2 P_n + 2 \left(-r - \frac{\ell}{r} \right) \partial_r P_n + 2nP_n = 0, \quad \lambda \in \Lambda. \end{aligned}$$

These polynomials exist for all $\lambda \in \Lambda$ when their degrees are $n \in 2\mathbb{Z}$, while for $n \in 2\mathbb{Z} + 1$, they exist only for the cases Λ_{-+} and Λ_{--} with special conditions. Hence we have obtained the algebraic spectrum $\Lambda = 2\mathbb{Z}$, where $\Lambda_{++} = 4\mathbb{Z}_-$, $\Lambda_{+-} = 2\mathbb{Z}_-$, $\Lambda_{-+} = 4\mathbb{Z}_+$, $\Lambda_{--} = 2\mathbb{Z}$.

The possibilities for eigenfunctions, considering only $\lambda \in 4\mathbb{Z}$, are given by

$$\begin{aligned} \Lambda_{++}) \quad & \Psi_n(r) = r^{\ell+1} P_{2n}(r) e^{\frac{r^2}{2}}, \quad \lambda \in \Lambda_-, \\ \Lambda_{+-}) \quad & \Psi_n(r) = r^{-\ell} P_{2n}(r) e^{\frac{r^2}{2}}, \quad \lambda \in \Lambda_-, \\ \Lambda_{-+}) \quad & \Psi_n(r) = r^{\ell+1} P_{2n}(r) e^{-\frac{r^2}{2}}, \quad \lambda \in \Lambda_+, \\ \Lambda_{--}) \quad & \Psi_n(r) = r^{-\ell} P_{2n}(r) e^{-\frac{r^2}{2}}, \quad \lambda \in \Lambda. \end{aligned}$$

To obtain the point spectrum, we look for Ψ_n satisfying the bound state conditions which is in only true for $\lambda \in \Lambda_{-+}$. With the change $r \mapsto \sqrt{\frac{\omega}{2}} r$, the point spectrum and ground state of the Schrödinger equation with the 3D-harmonic oscillator potential are respectively $\text{Spec}_p(H) = \{E_n : n \in \mathbb{Z}_+\}$, for $E_n = 2n\omega$, where ω is the angular velocity, and

$$\Psi_0 = \left(\sqrt{\frac{\omega}{2}} r \right)^{\ell+1} e^{-\frac{\omega}{4} r^2}.$$

The bound state wave functions are obtained as $\Psi_n = P_{2n} \Psi_0$. Now, we can see that $\text{DGal}(L_0/K) = \mathbb{B}$ and $\mathcal{E}(H) = \text{Vect}(1)$. Since $\Psi_n = P_{2n} \Psi_0$, for all $\lambda \in \Lambda$ we have $\text{DGal}(L_\lambda/K) = \mathbb{B}$ and $\mathcal{E}(H - \lambda) = \text{Vect}(1)$. In particular, $\text{DGal}(L_\lambda/K) = \mathbb{B}$ and $\mathcal{E}(H - \lambda) = \text{Vect}(1)$ for all $\lambda \in \text{Spec}_p(H)$, where $\lambda = \frac{2}{\omega} E$.

We remark that the Schrödinger equation with the 3D-harmonic oscillator potential, through the changes $r \mapsto \frac{1}{2}\omega r^2$ and $\Psi \mapsto \sqrt{r} \Psi$, fall in the class of Whittaker differential equations in which the parameters are given by

$$\kappa = \frac{(2\ell + 3)\omega + 2E}{4\omega}, \quad \mu = \frac{1}{2}\ell + \frac{1}{4}.$$

By Martinet-Ramis theorem, we have integrability when $\pm\kappa \pm \mu$ is a half integer. These conditions coincides with our four sets $\Lambda_{\pm\pm}$.

Coulomb potential:

$$V(r) = -\frac{e^2}{r} + \frac{\ell(\ell+1)}{r^2} + \frac{e^4}{4(\ell+1)^2}, \quad \ell \in \mathbb{Z}.$$

The Schrödinger equation in this case can be written as

$$\partial_r^2 \Psi = \left(\frac{\ell(\ell+1)}{r^2} - \frac{e^2}{r} + \frac{e^4}{4(\ell+1)^2} - E \right) \Psi.$$

By the change $r \mapsto \frac{2(\ell+1)}{e^2} r$ we obtain the Schrödinger equation

$$\partial_r^2 \Psi = \left(\frac{\ell(\ell+1)}{r^2} - \frac{2(\ell+1)}{r} + 1 - \lambda \right) \Psi, \quad \lambda = \frac{4(\ell+1)^2}{e^4} E$$

and in order to apply Kovacic algorithm, we set

$$R = \frac{\ell(\ell+1)}{r^2} - \frac{2(\ell+1)}{r} + 1 - \lambda.$$

First we analyze the case for $\lambda = 1$: we can see that this equation only could fall in case 2 or in case 4 of Kovacic's algorithm. We start discarding the case 2 because by step 1 we should have conditions c_2 and ∞_3 . So we should have $E_c = \{2, 4 + 4\ell, -4\ell\}$ and $E_\infty = \{1\}$ and by step 2, we would have $n \notin \mathbb{Z}$. Thus, $n \notin \mathbb{Z}_+$ and $D = \emptyset$, so the differential Galois group of this Schrödinger equation for $\lambda = 1$ is $\text{SL}(2, \mathbb{C})$.

Now, we analyze the case for $\lambda \neq 1$: we can see that this equation could fall in case 1, in case 2 or in case 4. We start discarding the case 2 because by step 1 we should have conditions c_2 and ∞_3 , so that we should have $E_c = \{2, 4 + 4\ell, -4\ell\}$ and $E_\infty = \{0\}$. By step 2, we should have $n = 2\ell \in \mathbb{Z}_+$, so that $D = \{2\ell\}$ and the rational function θ is $\theta = \frac{-2\ell}{x}$, but we discard this case because there could only exist one polynomial of degree 2ℓ for a fixed ℓ , and for instance, there could only exist one eigenstate and one eigenfunction for the Schrödinger equation.

Now, we only work with case 1, by step 1, conditions c_2 and ∞_3 are satisfied. Thus,

$$\left[\sqrt{R} \right]_c = 0, \quad \alpha_c^\pm = \frac{1 \pm (2\ell+1)}{2}, \quad \left[\sqrt{R} \right]_\infty = \sqrt{1-\lambda}, \quad \alpha_\infty^\pm = \mp \frac{\ell+1}{\sqrt{1-\lambda}}.$$

By step 2 we have the following possibilities for $n \in \mathbb{Z}_+$ and for $\lambda \in \Lambda$:

$$\Lambda_{++}) \quad n = \alpha_\infty^+ - \alpha_0^+ = -(\ell+1) \left(1 + \frac{1}{\sqrt{1-\lambda}} \right), \quad \lambda = 1 - \left(\frac{\ell+1}{\ell+1+n} \right)^2,$$

$$\Lambda_{+-}) \quad n = \alpha_\infty^+ - \alpha_0^- = -\frac{\ell+1}{\sqrt{1-\lambda}} + \ell, \quad \lambda = 1 - \left(\frac{\ell+1}{\ell-n} \right)^2,$$

$$\Lambda_{-+}) \quad n = \alpha_\infty^- - \alpha_0^+ = (\ell+1) \left(\frac{1}{\sqrt{1-\lambda}} - 1 \right), \quad \lambda = 1 - \left(\frac{\ell+1}{\ell+1+n} \right)^2,$$

$$\Lambda_{--}) \quad n = \alpha_\infty^- - \alpha_0^- = \frac{\ell+1}{\sqrt{1-\lambda}} + \ell, \quad \lambda = 1 - \left(\frac{\ell+1}{\ell-n} \right)^2.$$

We can see that $\lambda \in \Lambda_-$ when $\lambda \leq 0$, while $\lambda \in \Lambda_+$ when $0 \leq \lambda < 1$. Furthermore:

$$\Lambda_{++}) \quad \ell \leq -1, \quad \lambda \in \begin{cases} \Lambda_-, & \ell \leq \frac{n-2}{2} \\ \Lambda_+, & \frac{-n-2}{2} \leq \ell \leq -1 \end{cases}$$

$$\Lambda_{+-}) \quad \ell > 0, \quad \lambda \in \begin{cases} \Lambda_-, & \ell \geq \frac{n-1}{2} \\ \Lambda_+, & 0 \leq \ell \leq \frac{n-1}{2} \end{cases}$$

$$\Lambda_{-+}) \quad \ell \in \mathbb{Z}, \quad \lambda \in \begin{cases} \Lambda_-, & \ell \leq \frac{-n-2}{2} \\ \Lambda_+, & \ell \geq -1 \end{cases}$$

$$\Lambda_{--}) \quad \ell > 0, \quad \lambda \in \begin{cases} \Lambda_-, & \ell \geq \frac{n-1}{2} \\ \Lambda_+, & 0 \leq \ell \leq \frac{n-1}{2} \end{cases}$$

So the possible algebraic spectrum can be $\Lambda = \Lambda_{++} \cup \Lambda_{+-} \cup \Lambda_{-+} \cup \Lambda_{--}$, that is

$$(5.2) \quad \Lambda = \left\{ 1 - \left(\frac{\ell+1}{\ell+1+n} \right)^2 : n \in \mathbb{Z}_+ \right\} \cup \left\{ 1 - \left(\frac{\ell+1}{\ell-n} \right)^2 : n \in \mathbb{Z}_+ \right\},$$

Now, for $\lambda \in \Lambda$, the rational function ω is given by:

$$\Lambda_{++}) \quad \omega = \frac{\ell+1}{\ell+1+n} + \frac{\ell+1}{r}, \quad \lambda \in \Lambda_{++}, \quad R_n = \frac{\ell(\ell+1)}{r^2} - \frac{2(\ell+1)}{r} + \left(\frac{\ell+1}{\ell+1+n} \right)^2,$$

$$\Lambda_{+-}) \quad \omega = \frac{\ell+1}{\ell-n} - \frac{\ell}{r}, \quad \lambda \in \Lambda_{+-}, \quad R_n = \frac{\ell(\ell+1)}{r^2} - \frac{2(\ell+1)}{r} + \left(\frac{\ell+1}{\ell-n} \right)^2,$$

$$\Lambda_{-+}) \quad \omega = -\frac{\ell+1}{\ell+1+n} + \frac{\ell+1}{r}, \quad \lambda \in \Lambda_{-+}, \quad R_n = \frac{\ell(\ell+1)}{r^2} - \frac{2(\ell+1)}{r} + \left(\frac{\ell+1}{\ell+1+n} \right)^2,$$

$$\Lambda_{--}) \quad \omega = -\frac{\ell+1}{\ell-n} - \frac{\ell}{r}, \quad \lambda \in \Lambda_{--}, \quad R_n = \frac{\ell(\ell+1)}{r^2} - \frac{2(\ell+1)}{r} + \left(\frac{\ell+1}{\ell-n} \right)^2,$$

where R_n is the coefficient of the differential equation $\partial_r^2 \Psi = R_n \Psi$, which is integrable for every n .

By step 3, there exists a polynomial of degree n satisfying the relation (A.1),

$$\Lambda_{++}) \quad \partial_r^2 P_n + 2 \left(\frac{\ell+1}{\ell+1+n} + \frac{\ell+1}{r} \right) \partial_r P_n + \frac{2(\ell+1)}{r} \left(1 + \frac{\ell+1}{\ell+1+n} \right) P_n = 0,$$

$$\Lambda_{+-}) \quad \partial_r^2 P_n + 2 \left(\frac{\ell+1}{\ell-n} - \frac{\ell}{r} \right) \partial_r P_n + \frac{2(\ell+1)}{r} \left(1 - \frac{\ell+1}{\ell-n} \right) P_n = 0,$$

$$\Lambda_{-+}) \quad \partial_r^2 P_n + 2 \left(-\frac{\ell+1}{\ell+1+n} + \frac{\ell+1}{r} \right) \partial_r P_n + \frac{2(\ell+1)}{r} \left(1 - \frac{\ell+1}{\ell+1+n} \right) P_n = 0,$$

$$\Lambda_{--}) \quad \partial_r^2 P_n + 2 \left(-\frac{\ell+1}{\ell-n} - \frac{\ell}{r} \right) \partial_r P_n + \frac{2(\ell+1)}{r} \left(1 + \frac{\ell+1}{\ell-n} \right) P_n = 0.$$

These polynomials exist for every $\lambda \in \Lambda$ when $n \in \mathbb{Z}$, but $P_0 = 1$ is satisfied only for $\lambda \in \Lambda_{-+}$. Hence we have confirmed that the algebraic spectrum Λ is given by equation (5.2).

The possibilities for the eigenfunctions are given by

$$\Lambda_{++}) \quad \Psi_n(r) = r^{\ell+1} P_n(r) f_n(r) e^r, \quad f_n(r) = e^{\frac{-nr}{\ell+1+n}}, \quad \lambda \in \begin{cases} \Lambda_-, & \ell \leq \frac{-n-2}{2} \\ \Lambda_+, & \frac{-n-2}{2} \leq \ell \leq -1 \end{cases}$$

$$\Lambda_{+-}) \quad \Psi_n(r) = r^{-\ell} P_n(r) f_n(r) e^r, \quad f_n(r) = e^{\frac{n+1}{\ell-n}r}, \quad \lambda \in \begin{cases} \Lambda_-, & \ell \geq \frac{n-1}{2} \\ \Lambda_+, & 0 \leq \ell \leq \frac{n-1}{2} \end{cases}$$

$$\Lambda_{-+}) \quad \Psi_n(r) = r^{\ell+1} P_n(r) f_n(r) e^{-r}, \quad f_n(r) = e^{\frac{nr}{\ell+1+n}}, \quad \lambda \in \begin{cases} \Lambda_-, & \ell \leq \frac{-n-2}{2} \\ \Lambda_+, & \ell \geq -1 \end{cases}$$

$$\Lambda_{--}) \quad \Psi_n(r) = r^{-\ell} P_n(r) f_n(r) e^{-r}, \quad f_n(r) = e^{\frac{n+1}{n-\ell}r}, \quad \lambda \in \begin{cases} \Lambda_-, & \ell \geq \frac{n-1}{2} \\ \Lambda_+, & 0 \leq \ell \leq \frac{n-1}{2} \end{cases}$$

but Ψ_n should satisfy the bound state conditions which is only true for $\lambda \in \Lambda_{-+} \cap \Lambda_+$, so that we choose $\Lambda_{-+} \cap \Lambda_+ = \text{spec}_p(H)$, that is

$$\text{spec}_p(H) = \left\{ 1 - \left(\frac{\ell+1}{\ell+n+1} \right)^2 : n \in \mathbb{Z}_+, \ell \geq -1 \right\}.$$

By the change $r \mapsto \frac{e^2}{2(\ell+1)}r$, the point spectrum and ground state of the Schrödinger equation with Coulomb potential are respectively

$$\text{Spec}_p(H) = \{E_n : n \in \mathbb{Z}_+\}, \quad E_n = \frac{e^4}{4} \left(\frac{1}{(\ell+1)^2} - \frac{1}{(\ell+1+n)^2} \right)$$

and

$$\Psi_0 = \left(\frac{e^2}{2(\ell+1)}r \right)^{\ell+1} e^{-\frac{e^2}{2(\ell+1)}r}.$$

The eigenstates are given by $\Psi_n = P_n f_n \Psi_0$, where

$$f_n(r) = e^{\frac{ne^2r}{2(\ell+1+n)(\ell+1)}}.$$

Now, we can see that $\text{DGal}(L_0/K) = \mathbb{B}$ and $\mathcal{E}(H) = \text{Vect}(1)$. Since $\Psi_n = P_n f_n \Psi_0$, for all $\lambda \in \Lambda$ we have $\text{DGal}(L_\lambda/K) = \mathbb{B}$ and $\mathcal{E}(H - \lambda) = \text{Vect}(1)$. In particular, $\text{DGal}(L_\lambda/K) = \mathbb{B}$ and $\mathcal{E}(H - \lambda) = \text{Vect}(1)$ for all $E \in \text{Spec}_p(H)$, where $E = \frac{e^4}{4(\ell+1)^2}\lambda$.

We remark that, as in the three dimensional harmonic oscillator, the Schrödinger equation with the Coulomb potential, through the change

$$r \mapsto \frac{\sqrt{-4(\ell+1)^2 E + e^4}}{\ell+1} r,$$

falls in the class of Whittaker differential equation with the parameters given by

$$\kappa = \frac{e^2(\ell+1)}{\sqrt{-4(\ell+1)^2 E + e^4}}, \quad \mu = \ell + \frac{1}{2}.$$

Applying Martinet-Ramis theorem, we can impose $\pm\kappa \pm \mu$ half integer, to coincides with our four sets $\Lambda_{\pm\pm}$.

Remark 5.5. By direct application of Kovacic's algorithm we have:

- The Schrödinger equation (3.1) with potential

$$V(x) = ax^2 + \frac{b}{x^2}$$

is integrable for $\lambda \in \Lambda$ when

- $a = 0, \quad b = \mu(\mu + 1), \quad \mu \in \mathbb{C}, \quad \Lambda = \mathbb{C},$
- $a = 1, \quad b = 0, \quad \lambda \in \Lambda = 2\mathbb{Z} + 1,$
- $a = 1, \quad b = \ell(\ell + 1), \quad \ell \in \mathbb{Z}^*, \Lambda = 2\mathbb{Z} + 1.$
- The only rational potentials (up to transformations) in which the elements of the algebraic spectrum are placed at the same distance, belongs to the family of potentials given by

$$V(x) = \sum_{k=-\infty}^2 a_k x^k, \quad a_2 \neq 0.$$

In particular, the set Λ for the harmonic oscillator ($a = 1, b = 0$) and 3D harmonic oscillator ($a = 1, b = \ell(\ell + 1)$) satisfies this.

Proposition 5.6. *Let \mathcal{L}_λ be the Schrödinger equation (3.1) with $K = \mathbb{C}(x)$ and Picard-Vessiot extension L_λ . If $\text{DGal}(L_0/K)$ is finite primitive, then $\text{DGal}(L_\lambda/K)$ is not finite primitive for all $\lambda \in \Lambda \setminus \{0\}$.*

PROOF. Pick $V \in \mathbb{C}(x)$ such that \mathcal{L}_0 falls in case 3 of Kovacic algorithm, then $\circ u_\infty \geq 2$. Assume $t, s \in \mathbb{C}[x]$ such that $V = \frac{s}{t}$, then $\deg(t) \geq \deg(s) + 2$ and $V - \lambda = \frac{s - \lambda t}{t}$. Now, for $\lambda \neq 0$ we have $\deg(s - \lambda t) = \deg(t)$ and therefore $\circ(V - \lambda)_\infty = 0$. So, for $\lambda \neq 0$, the equation \mathcal{L}_λ does not falls in case 3 of Kovacic algorithm and therefore $\text{DGal}(L_\lambda/K)$ is not finite primitive. \square

Corollary 5.7. *Let \mathcal{L}_λ be the Schrödinger equation (3.1) with $K = \mathbb{C}(x)$ and Picard-Vessiot extension L_λ . If $\text{Card}(\Lambda) > 1$, then there is either zero or one value of λ such that $\text{DGal}(L_\lambda/K)$ is a finite primitive group.*

PROOF. Assume that $\text{Card}(\Lambda) > 1$. Thus, by proposition 5.6, the Schrödinger equation does not falls in case 3 of Kovacic's algorithm. \square

We note that the study of the differential Galois groups of the Schrödinger equation with the Coulomb potential had also been analyzed by Jean-Pierre Ramis ([51]) using his summability theory in the eighties of the past century.

6. Transcendental Potentials and the Algebrization Method

In supersymmetric quantum mechanics, there exists potentials which are not rational functions and, for this reason, it is difficult to apply our Galoisian approach from section 5. In this section, we address this problem and present some results concerning differential equations with non-rational coefficients. For these differential equations it is useful, whenever possible, to replace it by a new differential equation over the Riemann sphere \mathbb{P}^1 (that is, with rational coefficients). We will propose techniques to find such a change of variable and demonstrate, through

numerous examples, how our approach allows us to handle many examples from quantum mechanics. The equation over \mathbb{P}^1 is called an *algebraic form* or *algebrization* of the original equation.

This algebraic form is already studied in the 19th century literature (Liouville, Darboux), but the problem of obtaining an algebraic form (if it exists) of a given differential equation is in general not an easy task. Here we develop a new methodology using the concept of *Hamiltonian change of variables*. Such a change of variables allows us to compute the algebraic form of a large number of differential equations of different types. In particular, for second order linear differential equations, we can apply *Kovacic's algorithm* over the algebraic form to solve the original equation.

The following definition can be found in [9, 69, 70].

Definition 6.1 (Pullbacks of differential equations). Let $\mathfrak{L}_1 \in K_1[\partial_z]$ and $\mathfrak{L}_2 \in K_2[\partial_x]$ be differential operators. The expression $\mathfrak{L}_2 \otimes (\partial_x + v)$ refers to the operator whose solutions are the solutions of \mathfrak{L}_2 multiplied by the solution $e^{-\int v dx}$ of $\partial_x + v$.

- \mathfrak{L}_2 is a *proper pullback* of \mathfrak{L}_1 by means of $f \in K_2$ if the change of variable $z = f(x)$ changes \mathfrak{L}_1 into \mathfrak{L}_2 .
- \mathfrak{L}_2 is a *pullback* (also known as weak pullback) of \mathfrak{L}_1 by means of $f \in K_2$ if there exists $v \in K_2$ such that $\mathfrak{L}_2 \otimes (\partial_x + v)$ is a proper pullback of \mathfrak{L}_1 by means of f .

In case of compact Riemann surfaces, the geometric mechanism behind the algebrization is a ramified covering of compact Riemann surfaces, see [44, 43].

6.1. Second Order Linear Differential Equations. Some results presented in this subsection also can be found in [4, §2].

Proposition 6.2 (Change of the independent variable, [4]). *Let us consider the following equation, with coefficients in $\mathbb{C}(z)$:*

$$(6.1) \quad \mathcal{L}_z := \partial_z^2 y + a(z)\partial_z y + b(z)y = 0,$$

and $\mathbb{C}(z) \hookrightarrow L$ the corresponding Picard-Vessiot extension. Let (K, δ) be a differential field with \mathbb{C} as field of constants. Let $\theta \in K$ be a non-constant element. Then, by the change of variable $z = \theta(x)$, equation (6.1) is transformed in

$$(6.2) \quad \mathcal{L}_x := \partial_x^2 r + \left(a(\theta)\partial_x \theta - \frac{\partial_x^2 \theta}{\partial_x \theta} \right) \partial_x r + b(\theta)(\partial_x \theta)^2 r = 0, \quad \partial_x = \delta, \quad r = y \circ \theta.$$

Let $K_0 \subset K$ be the smallest differential field containing θ and \mathbb{C} . Then equation (6.2) is a differential equation with coefficients in K_0 . Let $K_0 \hookrightarrow L_0$ be the corresponding Picard-Vessiot extension. Assume that

$$\mathbb{C}(z) \rightarrow K_0, \quad z \mapsto \theta$$

is an algebraic extension, then

$$(\mathrm{DGal}(L_0/K_0))^0 = (\mathrm{DGal}(L/\mathbb{C}(z)))^0.$$

Proposition 6.3. *Assume that \mathcal{L}_x and \mathcal{L}_z are as in proposition 6.2. Let φ be the transformation given by*

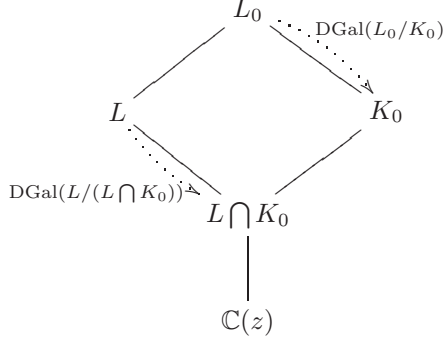
$$\varphi : \begin{cases} z \mapsto \theta(x) \\ \partial_z \mapsto \frac{1}{\partial_x \theta} \delta. \end{cases}$$

Then $\mathrm{DGal}(L_0/K_0) \simeq \mathrm{DGal}(L/K_0 \cap L) \subset \mathrm{DGal}(L/\mathbb{C}(z))$. Furthermore, if $K_0 \cap L$ is algebraic over $\mathbb{C}(z)$, then $(\mathrm{DGal}(L_0/K_0))^0 \simeq (\mathrm{DGal}(L/\mathbb{C}(z)))^0$.

PROOF. By Proposition 6.2, the transformation φ leads us to

$$\mathbb{C}(z) \simeq \varphi(\mathbb{C}(z)) \hookrightarrow K_0.$$

We identify $\mathbb{C}(z)$ with $\varphi(\mathbb{C}(z))$, so that we can view $\mathbb{C}(z)$ as a subfield of K_0 and then by the standard Galois theory diagram (see [33, 71]),



we have

$$\mathrm{DGal}(L_0/K_0) \simeq \mathrm{DGal}(L/K_0 \cap L) \subset \mathrm{DGal}(L/\mathbb{C}(z)).$$

If $K_0 \cap L$ is algebraic over $\mathbb{C}(z)$, then

$$(\mathrm{DGal}(L_0/K_0))^0 \simeq (\mathrm{DGal}(L/\mathbb{C}(z)))^0.$$

□

For the remainder of this section we write $z = z(x)$ instead of θ .

Remark 6.4 (Hard Algebrization). The proper pullback from equation (6.2) to equation (6.1) is an algebrization process. Therefore, we can try to algebrize any second order linear differential equations with non-rational coefficients (proper pull-back) if we can put it in the form of equation (6.2). To do this, which will be called *hard algebrization*, we use the following steps.

- Step 1:** Find a term of the form $(\partial_x z)^2$ in the coefficient of y to obtain a candidate for $\partial_x z$ and then z .
- Step 2:** Divide the coefficient of y by $(\partial_x z)^2$ to obtain a candidate $b(z)$ and check whether $b \in \mathbb{C}(z)$.
- Step 3:** Add $(\partial_x^2 z)/\partial_x z$ to the coefficient of $\partial_x y$ and divide the result by $\partial_x z$ to obtain a candidate $a(z)$ and check whether $a \in \mathbb{C}(z)$.

Let us illustrate this method of hard algebrization on the following example.

Example. In [58, p. 256], Singer presents the second order linear differential equation

$$\partial_x^2 r - \frac{1}{x(\ln x + 1)} \partial_x r - (\ln x + 1)^2 r = 0.$$

To algebrize this differential equation we choose $(\partial_x z)^2 = (\ln x + 1)^2$, so that $\partial_x z = \ln x + 1$ and for instance

$$z = \int (\ln x + 1) dx = x \ln x, \quad b(z) = -1.$$

Now we find $a(z)$ in the expression

$$a(z)(\ln x + 1) - \frac{1}{x(\ln x + 1)} = -\frac{1}{x(\ln x + 1)},$$

obtaining $a(z) = 0$. So that the new differential equation is given by $\partial_z^2 y - y = 0$, in which $y(z(x)) = r(x)$ and one basis of solutions of this differential equation is given by $\langle e^z, e^{-z} \rangle$. Thus, the respective basis of solutions of the first differential equation is given by $\langle e^{x \ln x}, e^{-x \ln x} \rangle$.

In general, this method is strongly heuristic because the quest of $z = z(x)$ in $b(z)(\partial_x z)^2$ can be purely a lottery, or simply there may not exist z such that $a(z), b(z) \in \mathbb{C}(z)$. For example, the equations presented by Singer in [58, p. 257, 261, 270] and given by

$$\partial_x^2 r + \frac{\mp 2x \ln^2 x \mp 2x \ln x - 1}{x \ln x + x} \partial_x r + \frac{-2x \ln^2 x - 3x \ln x - x \mp 1}{x \ln x + x} r = 0,$$

$$\partial_x^2 r + \frac{4x \ln x + 2x}{4x^2 \ln x} \partial_x r - \frac{1}{4x^2 \ln x} r = 0, \quad (x^2 \ln^2 x) \partial_x^2 r + (x \ln^2 x - 3x \ln x) \partial_x r + 3r = 0,$$

cannot be algebrized systematically with this method, although they correspond to pullbacks (not proper pullbacks) of differential equations with constant coefficients.

In [14], Bronstein and Fredet developed and implemented an algorithm to solve differential equations over $\mathbb{C}(x, e^{\int f})$ without algebrizing them, see also [26]. As an application of proposition 6.2 we have the following result¹.

Proposition 6.5 (Linear differential equation over $\mathbb{C}(x, e^{\int f})$, [4]). *Let $f \in \mathbb{C}(x)$ be a rational function. Then, the differential equation*

$$(6.3) \quad \partial_x^2 r - \left(f + \frac{\partial_x f}{f} - f e^{\int f} a(e^{\int f}) \right) \partial_x r + \left(f(e^{\int f}) \right)^2 b(e^{\int f}) r = 0,$$

is algebrizable by the change $z = e^{\int f}$ and its algebraic form is given by

$$\partial_z^2 y + a(z) \partial_z y + b(z) y = 0, \quad r(x) = y(z(x)).$$

PROOF. Assume that $r(x) = y(z(x))$, and $z = z(x) = e^{\int f dx}$. We can see that

$$\partial_x z = f z, \quad \partial_z y = \frac{\partial_x r}{f z}, \quad \partial_z^2 y = \frac{1}{f z} \partial_x \left(\frac{\partial_x r}{f z} \right) = \frac{1}{(f e^{\int f})^2} \left(\partial_x^2 r - f + \left(\frac{\partial_x f}{f} \right) \right) \partial_x r.$$

Replacing in $\partial_z^2 y + a(z) \partial_z y + b(z) y = 0$ we obtain equation (6.3). \square

¹This result is given in [4, §2], but we include here the proof for completeness.

Example. The differential equation

$$\partial_x^2 r - \left(x + \frac{1}{x} - 2xe^{x^2}\right) \partial_x r + \lambda x^2 e^{x^2} r = 0,$$

is algebrizable by the change $z = e^{\frac{x^2}{2}}$ and its algebraic form is given by

$$\partial_z^2 y + 2z \partial_z y + \lambda y = 0.$$

Remark 6.6. According to proposition 6.5, we have the following cases.

- (1) $f = n \frac{\partial_x h}{h}$, for a rational function h , $n \in \mathbb{Z}_+$, we have the trivial case, both equations are over the Riemann sphere and they have the same differential field, so that does not need to be algebrized.
- (2) $f = \frac{1}{n} \frac{\partial_x h}{h}$, for a rational function h , $n \in \mathbb{Z}^+$, (6.3) is defined over an algebraic extension of $\mathbb{C}(x)$ and so that this equation is not necessarily over the Riemann sphere.
- (3) $f \neq q \frac{\partial_x h}{h}$, for any rational function h , $q \in \mathbb{Q}$, (6.3) is defined over a transcendental extension of $\mathbb{C}(x)$ and so that this equation is not over the Riemann sphere.

To algebrize second order linear differential equations may be easier when the term in $\partial_x r$ is absent and the change of variable is *Hamiltonian*.

Definition 6.7 (Hamiltonian change of variable, [4]). A change of variable $z = z(x)$ is called *Hamiltonian* if $(z(x), \partial_x z(x))$ is a solution curve of the autonomous classical one degree of freedom Hamiltonian system

$$\begin{cases} \partial_x z = \partial_w H \\ \partial_x w = -\partial_z H \end{cases} \quad \text{with} \quad H = H(z, w) = \frac{w^2}{2} + V(z),$$

for some $V \in K$.

Remark 6.8. Assume that we algebrize equation (6.2) through a Hamiltonian change of variables $z = z(x)$, i.e. $V \in \mathbb{C}(z)$. Then $K_0 = \mathbb{C}(z, \partial_x z, \dots)$, but we have the algebraic relation

$$(\partial_x z)^2 = 2h - 2V(z), \quad h = H(z, \partial_x z) \in \mathbb{C},$$

so that $K_0 = \mathbb{C}(z, \partial_x z)$ is an algebraic extension of $\mathbb{C}(z)$. By proposition 6.2, the identity connected component of the differential Galois group is preserved. On the other hand, we can identify a Hamiltonian change of variable $z = z(x)$ when there exists $\alpha \in K$ such that $(\partial_x z)^2 = \alpha(z)$. Thus, we introduce the *Hamiltonian algebrization*, which corresponds to the algebrization process done through a Hamiltonian change of variable.

The following result, which also can be found in [4, §2], is an example of Hamiltonian algebrization and correspond to the case of reduced second order linear differential equations.

Proposition 6.9 (Hamiltonian Algebrization, [4]). *The differential equation*

$$\partial_x^2 r = q(x)r$$

is algebrizable through a Hamiltonian change of variable $z = z(x)$ if and only if there exist f, α such that

$$\frac{\partial_z \alpha}{\alpha}, \quad \frac{f}{\alpha} \in \mathbb{C}(z), \quad \text{where } f(z(x)) = q(x), \quad \alpha(z) = 2(H - V(z)) = (\partial_x z)^2.$$

Furthermore, the algebraic form of the equation $\partial_x^2 r = q(x)r$ is

$$(6.4) \quad \partial_z^2 y + \frac{1}{2} \frac{\partial_z \alpha}{\alpha} \partial_z y - \frac{f}{\alpha} y = 0, \quad r(x) = y(z(x)).$$

Remark 6.10 (Using the Algebrization Method). The goal is to algebrize the differential equation $\partial_x^2 r = q(x)r$. We propose the following steps.

Step 1: Look for a *Hamiltonian change of variable* $z = z(x)$ and two functions f and α such that $q(x) = f(z(x))$ and $(\partial_x z(x))^2 = \alpha(z(x))$.

Step 2: Verify whether or not $f(z)/\alpha(z) \in \mathbb{C}(z)$ and $\partial_z \alpha(z)/\alpha(z) \in \mathbb{C}(z)$ to see if the equation $\partial_x^2 r = q(x)r$ is algebrizable.

Step 3: If the equation $\partial_x^2 r = q(x)r$ is algebrizable, its algebrization is

$$\partial_z^2 y + \frac{1}{2} \frac{\partial_z \alpha}{\alpha} \partial_z y - \frac{f}{\alpha} y = 0, \quad y(z(x)) = r(x).$$

When we have algebrized the differential equation $\partial_x^2 r = q(x)r$, we study its integrability, its eigenring and its differential Galois group.

Examples. Consider the following examples.

- Given the differential equation $\partial_x^2 r = f(\tan x)r$ with $f \in \mathbb{C}(\tan x)$, we can choose $z = z(x) = \tan x$ to obtain $\alpha(z) = (1 + z^2)^2$, so that $z = z(x)$ is a Hamiltonian change of variable. We can see that $\frac{\partial_z \alpha}{\alpha}, \frac{f}{\alpha} \in \mathbb{C}(z)$ and the algebraic form of the differential equation $\partial_x^2 r = f(\tan x)r$ with this Hamiltonian change of variable is

$$\partial_z^2 y + \frac{2z}{1 + z^2} \partial_z y - \frac{f(z)}{(1 + z^2)^2} y = 0, \quad y(\tan x) = r(x).$$

- Given the differential equation

$$\partial_x^2 r = \frac{\sqrt{1 + x^2} + x^2}{1 + x^2} r,$$

we can choose $z = z(x) = \sqrt{1 + x^2}$ to obtain

$$f(z) = \frac{z^2 + z - 1}{z^2}, \quad \alpha(z) = \frac{z^2 - 1}{z^2},$$

so that $z = z(x)$ is a Hamiltonian change of variable. We can see that $\frac{\partial_z \alpha}{\alpha}, \frac{f}{\alpha} \in \mathbb{C}(z)$ and the algebraic form for this case is

$$\partial_z^2 y + \frac{1}{z(z^2 - 1)} \partial_z y - \frac{z^2 + z - 1}{z^2 - 1} y = 0, \quad y(\sqrt{1 + x^2}) = r(x).$$

We remark that in general the method of Hamiltonian algebrization is not an algorithm, because it may not be obvious to obtain a suitable Hamiltonian H satisfying definition 6.7. We present now a particular case of Hamiltonian algebrization considered as an algorithm².

Proposition 6.11 (Hamiltonian Algebrization Algorithm, [4]). *Consider $q(x) = g(z_1, \dots, z_n)$, where $z_i = e^{\lambda_i x}$, $\lambda_i \in \mathbb{C}^*$. The equation $\partial_x^2 r = q(x)r$ is algebrizable if and only if.*

$$\frac{\lambda_i}{\lambda_j} \in \mathbb{Q}^*, \quad 1 \leq i \leq n, 1 \leq j \leq n, \quad g \in \mathbb{C}(z).$$

²Proposition 6.11 is a slight improvement of a similar result given in [4, §2]. Furthermore, we include the proof here for completeness.

Furthermore, $\lambda_i = c_i \lambda$, where $\lambda \in \mathbb{C}^*$ and $c_i \in \mathbb{Q}^*$ and for the Hamiltonian change of variable

$$z = e^{\frac{\lambda x}{q}}, \text{ where } c_i = \frac{p_i}{q_i}, p_i, q_i \in \mathbb{Z}^*, \gcd(p_i, q_i) = 1 \text{ and } q = \text{lcm}(q_1, \dots, q_n),$$

the algebrization of the differential equation $\partial_x^2 r = q(x)r$ is

$$\partial_z^2 y + \frac{1}{z} \partial_z y - q^2 \frac{g(z^{m_1}, \dots, z^{m_n})}{\lambda^2 z^2} y = 0, \quad m_i = \frac{qp_i}{q_i}, \quad y(z(x)) = r(x).$$

PROOF. Assuming $\lambda_i/\lambda_j = c_{ij} \in \mathbb{Q}^*$ we can see that there exists $\lambda \in \mathbb{C}^*$ and $c_i \in \mathbb{Q}^*$ such that $\lambda_i = \lambda c_i$, so that

$$e^{\lambda_i x} = e^{c_i \lambda x} = e^{\frac{p_i}{q_i} \lambda x} = \left(e^{\frac{\lambda}{q} x} \right)^{\frac{qp_i}{q_i}}, \quad p_i, q_i \in \mathbb{Z}^*, \gcd(p_i, q_i) = 1, \text{lcm}(q_1, \dots, q_n) = q.$$

Now, setting $z = z(x) = e^{\frac{\lambda}{q} x}$ we can see that

$$f(z) = g(z^{m_1}, \dots, z^{m_n}), \quad m_i = \frac{qp_i}{q_i}, \quad \alpha = \frac{\lambda^2 z^2}{q^2}.$$

As $q|q_i$, we have $m_i \in \mathbb{Z}$, so that

$$\frac{\partial_z \alpha}{\alpha}, \quad \frac{f}{\alpha} \in \mathbb{C}(z)$$

and the algebraic form is given by

$$\partial_z^2 y + \frac{1}{z} \partial_z y - q^2 \frac{g(z^{m_1}, \dots, z^{m_n})}{\lambda^2 z^2} y = 0, \quad y(z(x)) = r(x).$$

□

Remark 6.12. Propositions 6.9 and 6.11 allow the algebrization of a large number of second order differential equations, see for example [49]. In particular, under the assumptions of proposition 6.11, we can algebrize automatically differential equations with trigonometrical or hyperbolic coefficients.

Examples. Consider the following examples.

- Given the differential equation

$$\partial_x^2 r = \frac{e^{\frac{1}{2}x} + 3e^{-\frac{2}{3}x} - 2e^{\frac{5}{4}x}}{e^x + e^{-\frac{3}{2}x}} r, \quad \lambda_1 = \frac{1}{2}, \lambda_2 = -\frac{2}{3}, \lambda_3 = \frac{5}{4}, \lambda_4 = 1, \lambda_5 = -\frac{3}{2},$$

we see that $\lambda_i/\lambda_j \in \mathbb{Q}$, $\lambda = 1$, $q = \text{lcm}(1, 2, 3, 4) = 12$ and the Hamiltonian change of variable for this case is $z = z(x) = e^{\frac{1}{12}x}$. We can see that

$$\alpha(z) = \frac{1}{144} z^2, \quad f(z) = \frac{z^6 + 3z^{-8} - 2z^{15}}{z^{12} + z^{-18}}, \quad \frac{\partial_z \alpha}{\alpha}, \frac{f}{\alpha} \in \mathbb{C}(z)$$

and the algebraic form is given by

$$\partial_z^2 y + \frac{1}{z} \partial_z y - 144 \frac{z^6 + 3z^{-8} - 2z^{15}}{z^{14} + z^{-16}} y = 0, \quad y(e^{\frac{1}{12}x}) = r(x).$$

- Given the differential equation

$$\partial_x^2 r = (e^{2\sqrt{2}x} + e^{-\sqrt{2}x} - e^{3x})r, \quad \lambda_1 = 2\sqrt{2}, \lambda_2 = -\sqrt{2}, \lambda_3 = 3,$$

we see that $\lambda_1/\lambda_2 \in \mathbb{Q}$, but $\lambda_1/\lambda_3 \notin \mathbb{Q}$, so that this differential equation cannot be algebrized.

We remark that it is also possible to use the algebrization method to transform differential equations when either the starting equation has rational coefficients or the transformed equation does not have rational coefficients.

Examples. As an illustration, we treat the following examples.

- Consider the following differential equation:

$$\partial_x^2 r = \frac{x^4 + 3x^2 - 5}{x^2 + 1} r = 0.$$

We can choose $z = z(x) = x^2$ so that $\alpha = 4z$ and the new differential equation is

$$\partial_z^2 y + \frac{1}{2z} \partial_z y - \frac{z^2 + 2z - 5}{4z(z + 1)} y = 0, \quad y(x^2) = r(x).$$

- Consider the Mathieu differential equation $\partial_x^2 r = (a + b \cos(x))r$. We can choose $z(x) = \ln(\cos(x))$ so that $\alpha = e^{-2z} - 1$ and the new differential equation is

$$\partial_z^2 y - \frac{1}{1 - e^{2z}} \partial_z y - \frac{ae^{2z} + be^{3z}}{1 - e^{2z}} y = 0, \quad y(\ln(\cos(x))) = r(x).$$

Recently, the Hamiltonian algebrization (propositions 6.9 and 6.11) has been applied in [1, 2, 4] to obtain non-integrability results.

6.2. The Operator $\hat{\partial}_z$ and the Hamiltonian Algebrization. The generalization of proposition 6.2 to higher order linear differential equations is difficult. But, it is possible to obtain generalizations of proposition 6.9 by means of Hamiltonian change of variable. We recall that $z = z(x)$ is a Hamiltonian change of variable if there exists α such that $(\partial_x z)^2 = \alpha(z)$. More specifically, if $z = z(x)$ is a Hamiltonian change of variable, we can write $\partial_x z = \sqrt{\alpha}$, which leads us to the following notation: $\hat{\partial}_z = \sqrt{\alpha} \partial_z$.

We can see that $\hat{\partial}_z$ is a *derivation* because satisfy $\hat{\partial}_z(f + g) = \hat{\partial}_z f + \hat{\partial}_z g$ and the Leibnitz rules

$$\hat{\partial}_z(f \cdot g) = \hat{\partial}_z f \cdot g + f \cdot \hat{\partial}_z g, \quad \hat{\partial}_z \left(\frac{f}{g} \right) = \frac{\hat{\partial}_z f \cdot g - f \cdot \hat{\partial}_z g}{g^2}.$$

We can notice that the chain rule is given by $\hat{\partial}_z(f \circ g) = \partial_g f \circ g \hat{\partial}_z(g) \neq \hat{\partial}_g f \circ g \hat{\partial}_z(g)$. The iteration of $\hat{\partial}_z$ is given by

$$\hat{\partial}_z^0 = 1, \quad \hat{\partial}_z = \sqrt{\alpha} \partial_z, \quad \hat{\partial}_z^n = \sqrt{\alpha} \partial_z \hat{\partial}_z^{n-1} = \underbrace{\sqrt{\alpha} \partial_z (\dots (\sqrt{\alpha} \partial_z))}_{n \text{ times } \sqrt{\alpha} \partial_z}.$$

We say that a Hamiltonian change of variable is rational when the potential $V \in \mathbb{C}(x)$ and for instance $\alpha \in \mathbb{C}(x)$. For the remainder of this paper, we let $\hat{\partial}_z = \sqrt{\alpha} \partial_z$ where $z = z(x)$ is a Hamiltonian change of variable and $\partial_x z = \sqrt{\alpha}$. In particular, $\hat{\partial}_z = \partial_z = \partial_x$ if and only if $\sqrt{\alpha} = 1$, i.e., $z = x$.

Theorem 6.13. *Consider the systems of linear differential equations $[A]$ and $[\hat{A}]$ given respectively by*

$$\partial_x \mathbf{Y} = -A\mathbf{Y}, \quad \hat{\partial}_z \hat{\mathbf{Y}} = -\hat{A}\hat{\mathbf{Y}}, \quad A = [a_{ij}], \quad \hat{A} = [\hat{a}_{ij}], \quad \mathbf{Y} = [y_{i1}], \quad \hat{\mathbf{Y}} = [\hat{y}_{i1}],$$

where $a_{ij} \in K = \mathbb{C}(z(x), \partial_x(z(x)))$, $\hat{a}_{ij} \in \mathbb{C}(z) \subseteq \hat{K} = \mathbb{C}(z, \sqrt{\alpha})$, $1 \leq i \leq n$, $1 \leq j \leq n$, $a_{ij}(x) = \hat{a}_{ij}(z(x))$ and $y_{i1}(x) = y_{i1}(z(x))$. Suppose that L and \hat{L} are the Picard-Vessiot extensions of $[A]$ and $[\hat{A}]$ respectively. If the transformation φ is given by

$$\varphi : \begin{cases} x \mapsto z \\ a_{ij} \mapsto \hat{a}_{ij} \\ y_{i1}(x) \mapsto \hat{y}_{i1}(z(x)) \\ \partial_x \mapsto \hat{\partial}_z \end{cases} ,$$

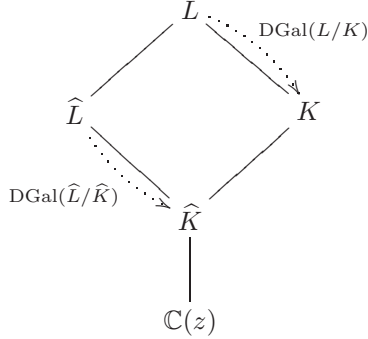
then the following statements hold.

- $K \simeq \hat{K}$, $(K, \partial_x) \simeq (\hat{K}, \hat{\partial}_z)$.
- $\text{DGal}(L/K) \simeq \text{DGal}(\hat{L}/\hat{K}) \subset \text{DGal}(\hat{L}/\mathbb{C}(z))$.
- $(\text{DGal}(L/K))^0 \simeq (\text{DGal}(\hat{L}/\mathbb{C}(z)))^0$.
- $\mathcal{E}(A) \simeq \mathcal{E}(\hat{A})$.

PROOF. We proceed as in the proof of proposition 6.3. As $z = z(x)$ is a rational Hamiltonian change of variable, the transformation φ leads us to

$$\mathbb{C}(z) \simeq \varphi(\mathbb{C}(z)) \hookrightarrow K, \quad K \simeq \hat{K}, \quad \mathbb{C}(z) \hookrightarrow \hat{K}, \quad (K, \partial_x) \simeq (\hat{K}, \hat{\partial}_z)$$

that is, we identify $\mathbb{C}(z)$ with $\varphi(\mathbb{C}(z))$, and so that we can view $\mathbb{C}(z)$ as a subfield of K and then, by the standard Galois theory diagram (see [33, 71]),



so that we have $\text{DGal}(L/K) \simeq \text{DGal}(\hat{L}/\hat{K}) \subset \text{DGal}(\hat{L}/\mathbb{C}(z))$, $(\text{DGal}(L/K))^0 \simeq (\text{DGal}(\hat{L}/\mathbb{C}(z)))^0$, and $\mathcal{E}(A) \simeq \mathcal{E}(\hat{A})$. \square

We remark that the transformation φ , given in theorem 6.13, is virtually strong isogaloisian when $\sqrt{\alpha} \notin \mathbb{C}(z)$ and for $\sqrt{\alpha} \in \mathbb{C}(z)$, φ is strong isogaloisian. Furthermore, by cyclic vector method (see [65]), we can write the systems $[A]$ and $[\hat{A}]$ in terms of the differential equations \mathcal{L} and $\hat{\mathcal{L}}$. Thus, $\hat{\mathcal{L}}$ is the proper pullback of \mathcal{L} and $\mathcal{E}(\mathfrak{L}) \simeq \mathcal{E}(\hat{\mathfrak{L}})$.

Example. Consider the system

$$[A] := \begin{cases} \partial_x \gamma_1 = -\frac{2\sqrt{2}}{e^x + e^{-x}} \gamma_3, \\ \partial_x \gamma_2 = \frac{e^x - e^{-x}}{e^x + e^{-x}} \gamma_3, \\ \partial_x \gamma_3 = \frac{2\sqrt{2}}{e^x + e^{-x}} \gamma_1 - \frac{e^x - e^{-x}}{e^x + e^{-x}} \gamma_2 \end{cases}$$

Through the Hamiltonian change of variable $z = e^x$, and with $\sqrt{\alpha} = z$, it is transformed into the system

$$[\widehat{A}] := \begin{cases} \partial_z \widehat{\gamma}_1 = -\frac{2\sqrt{2}}{z^2+1} \widehat{\gamma}_3, \\ \partial_z \widehat{\gamma}_2 = \frac{z^2-1}{z(z^2+1)} \widehat{\gamma}_3, \\ \partial_z \widehat{\gamma}_3 = \frac{2\sqrt{2}}{z^2+1} \widehat{\gamma}_1 - \frac{z^2-1}{x(x^2+1)} \widehat{\gamma}_2 \end{cases}.$$

One solution of the system $[\widehat{A}]$ is given by

$$\frac{1}{z^2+1} \begin{pmatrix} \frac{\sqrt{2}}{2}(1-z^2) \\ z \\ -z \end{pmatrix},$$

and the corresponding solution for the system $[A]$ is

$$\frac{1}{e^{2x}+1} \begin{pmatrix} \frac{\sqrt{2}}{2}(1-e^{2x}) \\ e^x \\ -e^x \end{pmatrix}.$$

Remark 6.14. The algebrization given in proposition 6.9 is an example of how the introduction of the new derivative $\widehat{\partial}_z$ simplifies the proofs and computations. Such proposition is naturally extended to $\partial_x^2 y + a\partial_x y + by = 0$, using φ to obtain $\widehat{\partial}_z^2 \widehat{y} + \widehat{a}\widehat{\partial}_z \widehat{y} + \widehat{b}\widehat{y} = 0$, which is equivalent to

$$(6.5) \quad \alpha \partial_z^2 \widehat{y} + \left(\frac{\partial_x \alpha}{2} + \sqrt{\alpha} \widehat{a} \right) \partial_z \widehat{y} + \widehat{b} \widehat{y} = 0,$$

where $y(x) = \widehat{y}(z(x))$, $\widehat{a}(z(x)) = a(x)$ and $\widehat{b}(z(x)) = b(x)$.

In general, for $y(x) = \widehat{y}(z(x))$, the equation $F(\partial_x^n y, \dots, y, x) = 0$ with coefficients given by $a_{i_k}(x)$ is transformed into the equation $\widehat{F}(\widehat{\partial}_z^n \widehat{y}, \dots, \widehat{y}, z) = 0$ with coefficients given by $\widehat{a}_{i_k}(z)$, where $a_{i_k}(x) = \widehat{a}_{i_k}(z(x))$. In particular, for $\sqrt{\alpha}, \widehat{a}_{i_k} \in \mathbb{C}(z)$, the equation $\widehat{F}(\widehat{\partial}_z^n \widehat{y}, \dots, \widehat{y}, z) = 0$ is the Hamiltonian algebrization of $F(\partial_x^n y, \dots, y, x) = 0$. Now, if each derivation ∂_x has even order, then α and \widehat{a}_{i_k} can be rational functions when we algebrize the equation $F(\partial_x^n y, \dots, y, x) = 0$, where $a_{i_k} \in \mathbb{C}(z(x), \partial_x z(x))$. For example, that happens for linear differential equations given by

$$\partial_x^{2n} y + a_{n-1}(x) \partial_x^{2n-2} y + \dots + a_2(x) \partial_x^4 y + a_1(x) \partial_x^2 y + a_0(x) y = 0.$$

Finally, the algebrization algorithm given in proposition 6.11 can be naturally extended to a differential equation of the form

$$F(\partial_x^n y, \partial_x^{n-1} y, \dots, \partial_x y, y, e^{\mu t}) = 0,$$

which, through the change of variable $z = e^{\mu x}$, is transformed into

$$\widehat{F}(\widehat{\partial}_z^n \widehat{y}, \widehat{\partial}_z^{n-1} \widehat{y}, \dots, \widehat{\partial}_z \widehat{y}, \widehat{y}, z) = 0.$$

In particular, we may consider the algebrization of Riccati equations, higher order linear differential equations and systems.

Examples. The following corresponds to some examples of algebrizations for differential equations given in [58, p. 258, 266].

- (1) The equation $\mathcal{L} := \partial_x^2 y + (-2e^x - 1)\partial_x y + e^{2x}y = 0$ with the Hamiltonian change of variable $z = e^x$, $\sqrt{\alpha} = z$, $\hat{a} = -2z - 1$ and $\hat{b} = z^2$ is transformed into the equation $\hat{\mathcal{L}} := \partial_z^2 \hat{y} - 2\partial_z \hat{y} + \hat{y} = 0$ which can be easily solved. Bases of solutions for \mathcal{L} and $\hat{\mathcal{L}}$ are respectively given by $\langle e^z, ze^z \rangle$ and $\langle e^{e^x}, e^x e^{e^x} \rangle$. Furthermore $K = \mathbb{C}(e^x)$, $\hat{K} = \mathbb{C}(z)$, L and \hat{L} are the Picard-Vessiot extensions of \mathcal{L} and $\hat{\mathcal{L}}$ respectively. Thus, $\text{DGal}(L/K) = \text{DGal}(\hat{L}/\hat{K})$.
- (2) The differential equation

$$\mathcal{L} := \partial_x^2 y + \frac{-24e^x - 25}{4e^x + 5}\partial_x y + \frac{20e^x}{4e^x + 5}y = 0$$

with the Hamiltonian change of variable $z = e^x$, $\sqrt{\alpha} = z$,

$$\hat{a} = \frac{-24z - 25}{4z + 5} \text{ and } \hat{b} = \frac{20z}{4z + 5}$$

is transformed into the equation

$$\hat{\mathcal{L}} := \partial_z^2 \hat{y} + \frac{-20(z+1)}{z(4z+5)}\partial_z \hat{y} + \frac{20}{z(4z+5)}\hat{y} = 0,$$

which can be solved with Kovacic algorithm. A basis of solutions for $\hat{\mathcal{L}}$ is $\langle z+1, z^5 \rangle$, so that a basis for \mathcal{L} is $\langle e^x + 1, e^{5x} \rangle$. Furthermore $K = \mathbb{C}(e^x)$, $\hat{K} = \mathbb{C}(z)$, L and \hat{L} are the Picard-Vessiot extensions of \mathcal{L} and $\hat{\mathcal{L}}$ respectively. Thus, $\text{DGal}(L/K) = \text{DGal}(\hat{L}/\hat{K}) = e$.

Remark 6.15 (Algebrization of the Riccati equation). The Riccati equation

$$(6.6) \quad \partial_x v = a(x) + b(x)v + c(x)v^2.$$

Through the Hamiltonian change of variable $z = z(x)$, it is transformed into the Riccati equation

$$(6.7) \quad \partial_z \hat{v} = \frac{1}{\sqrt{\alpha}}(\hat{a}(z) + \hat{b}(z)\hat{v} + \hat{c}(z)\hat{v}^2),$$

where $v(x) = \hat{v}(z(x))$, $\hat{a}(z(x)) = a(x)$, $\hat{b}(z(x)) = b(x)$, $\hat{c}(z(x)) = c(x)$ and $\sqrt{\alpha(z(x))} = \partial_x z(x)$. Furthermore, if $\sqrt{\alpha}$, \hat{a} , \hat{b} , $\hat{c} \in \mathbb{C}(x)$, equation (6.7) is an algebrization of equation (6.6).

Example. Consider the Riccati differential equation

$$\mathcal{L} := \partial_x v = \left(\tanh x - \frac{1}{\tanh x} \right) v + (3 \tanh x - 3 \tanh^3 x) v^2,$$

which through the Hamiltonian change of variable $z = \tanh x$, for instance $\sqrt{\alpha} = 1 - z^2$, is transformed into the Riccati differential equation

$$\hat{\mathcal{L}} := \partial_z v = -\frac{1}{z}v + 3zv^2.$$

One solution for the equation $\hat{\mathcal{L}}$ is

$$-\frac{1}{z(3z-c)}, \text{ where } c \text{ is a constant,}$$

so that the corresponding solution for equation \mathcal{L} is

$$-\frac{1}{\tanh x(3 \tanh x - c)}.$$

The following result is the algebrized version of the relationship between the eigenring s of systems and operators.

Proposition 6.16. *Consider the differential fields K, \widehat{K} and consider the systems $[A]$ and $[\widehat{A}]$ given by*

$$\partial_x \mathbf{X} = -A\mathbf{X}, \quad \partial_z \widehat{\mathbf{X}} = -\widehat{A}\widehat{\mathbf{X}}, \quad \partial_z = \sqrt{\alpha} \partial_x, \quad A = [a_{ij}], \quad \widehat{A} = [\widehat{a}_{ij}], \quad a_{ij} \in K, \quad \widehat{a}_{ij} \in \widehat{K},$$

where $z = z(x)$, $\mathbf{X}(x) = \widehat{\mathbf{X}}(z(x))$, $\widehat{a}_{ij}(z(x)) = a_{ij}(x)$, then their eigenrings satisfy $\mathcal{E}(A) \simeq \mathcal{E}(\widehat{A})$. In particular, if we consider the linear differential equations

$$\mathcal{L} := \partial_x^n y + \sum_{k=0}^{n-1} a_k \partial_x^k y = 0 \quad \text{and} \quad \widehat{\mathcal{L}} := \partial_z^n \widehat{y} + \sum_{k=0}^{n-1} \widehat{a}_k \partial_z^k \widehat{y} = 0,$$

where $z = z(x)$, $y(x) = \widehat{y}(z(x))$, $\widehat{a}_k(z(x)) = a_k(x)$, $a_k \in K$, $\widehat{a}_k \in \widehat{K}$, then $\mathcal{E}(\mathcal{L}) \simeq \mathcal{E}(\widehat{\mathcal{L}})$, where $\mathcal{L} := \mathfrak{L}y = 0$ and $\widehat{\mathcal{L}} := \widehat{\mathfrak{L}}\widehat{y} = 0$. Furthermore, assuming

$$P = \begin{pmatrix} p_{11} & \cdots & p_{1n} \\ \vdots & & \\ p_{n1} & \cdots & p_{nn} \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & & & \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{pmatrix},$$

then

$$\mathcal{E}(\mathfrak{L}) = \left\{ \sum_{k=1}^n p_{1k} \partial_x^{k-1} : \partial_x P = PA - AP, p_{ik} \in K \right\},$$

if and only if

$$\mathcal{E}(\widehat{\mathfrak{L}}) = \left\{ \sum_{k=1}^n \widehat{p}_{1k} \partial_z^{k-1} : \partial_z \widehat{P} = \widehat{P}\widehat{A} - \widehat{A}\widehat{P}, \widehat{p}_{ik} \in \widehat{K} \right\}.$$

PROOF. By theorem 6.13 we have $K \simeq \widehat{K}$, $\mathcal{E}(A) \simeq \mathcal{E}(\widehat{A})$ and $\mathcal{E}(\mathfrak{L}) \simeq \mathcal{E}(\widehat{\mathfrak{L}})$. Using the derivation ∂_z and by induction on lemma 6.3 we complete the proof. \square

Examples. We consider two different examples to illustrate the previous proposition.

- Consider the differential equation $\mathcal{L}_1 := \partial_x^2 y - (1 + \cos x - \cos^2 x)y = 0$. By means of the Hamiltonian change of variable $z = z(x) = \cos x$, with $\sqrt{\alpha} = -\sqrt{1 - z^2}$, \mathcal{L}_1 is transformed into the differential equation

$$\widehat{\mathcal{L}}_1 := \partial_z^2 \widehat{y} - \frac{z}{1 - z^2} \partial_z \widehat{y} - \frac{1 + z - z^2}{1 - z^2} \widehat{y} = 0.$$

Now, computing the eigenring of $\widehat{\mathfrak{L}}_1$ we have $\mathcal{E}(\widehat{\mathfrak{L}}_1) = \text{Vect}(1)$, therefore the eigenring of \mathfrak{L}_1 is given $\mathcal{E}(\mathfrak{L}_1) = \text{Vect}(1)$.

- Now we consider the differential equation $\mathcal{L}_2 := \partial_x^2 y = (e^{2x} + \frac{9}{4})y$. By means of the Hamiltonian change of variable $z = e^x$, with $\sqrt{\alpha} = x$, \mathcal{L}_2 is transformed into the differential equation

$$\widehat{\mathcal{L}}_2 := \partial_z^2 \widehat{y} + \frac{1}{z} \partial_z \widehat{y} - \left(1 + \frac{9}{4x^2}\right) \widehat{y} = 0.$$

Now, computing the eigenring of $\widehat{\mathfrak{L}}_2$ we have

$$\mathcal{E}(\widehat{\mathfrak{L}}_2) = \text{Vect} \left(1, -2 \left(\frac{z^2-1}{z^2} \right) \partial_z - \frac{z^2-3}{z^3} \right) = \text{Vect} \left(1, -2 \left(\frac{z^2-1}{z^3} \right) \widehat{\partial}_z - \frac{z^2-3}{z^3} \right),$$

therefore the eigenring of \mathfrak{L}_2 is given by

$$\mathcal{E}(\mathfrak{L}_2) = \text{Vect} \left(1, -2 \left(\frac{e^{2x}-1}{e^{3x}} \right) \partial_x - \frac{e^{2x}-3}{e^{3x}} \right).$$

The same result is obtained via matrix formalism, where

$$A = \begin{pmatrix} 0 & 1 \\ e^{2x} + \frac{9}{4} & 0 \end{pmatrix}, \quad \widehat{A} = \begin{pmatrix} 0 & 1 \\ z^2 + \frac{9}{4} & 0 \end{pmatrix}, \quad \partial_x P = PA - AP, \quad \widehat{\partial}_z \widehat{P} = \widehat{P} \widehat{A} - \widehat{A} \widehat{P},$$

with $P \in \text{Mat}(2, \mathbb{C}(e^x))$ and $\widehat{P} \in \text{Mat}(2, \mathbb{C}(z))$.

6.3. Applications in Supersymmetric Quantum Mechanics. In this subsection we apply the derivation $\widehat{\partial}_z$ to the Schrödinger equation $H\Psi = \lambda\Psi$, where $H = -\partial_x^2 + V(x)$, $V \in K$. Assume that $z = z(x)$ is a rational Hamiltonian change of variable for $H\Psi = \lambda\Psi$, then $K = \mathbb{C}(z(x), \partial_x z(x))$. Thus, the *algebrized Schrödinger equation* is written as

$$(6.8) \quad \widehat{H}\widehat{\Psi} = \lambda\widehat{\Psi}, \quad \widehat{H} = -\widehat{\partial}_z^2 + \widehat{V}(z), \quad \widehat{\partial}_z^2 = \alpha\partial_z^2 + \frac{1}{2}\partial_z\alpha\partial_z, \quad \widehat{K} = \mathbb{C}(z, \sqrt{\alpha}).$$

The *reduced algebrized Schrödinger equation* is given by

$$(6.9) \quad \widehat{\mathbf{H}}\Phi = \lambda\Phi, \quad \widehat{\mathbf{H}} = \alpha(z) \left(-\partial_z^2 + \widehat{\mathbf{V}}(z) \right), \quad \mathcal{V} = \partial_z \mathcal{W} + \mathcal{W}^2, \\ \mathcal{W} = \frac{1}{4} \frac{\partial_z \alpha(z)}{\alpha(z)}.$$

The eigenfunctions Ψ , $\widehat{\Psi}$ and Φ corresponding to the operators H , \widehat{H} and $\widehat{\mathbf{H}}$ are related respectively by

$$\Phi(z(x)) = \sqrt[4]{\alpha} \widehat{\Psi}(z(x)) = \sqrt[4]{\alpha} \Psi(x).$$

In order to apply the Kovacic's algorithm, we only consider the algebrized operator $\widehat{\mathbf{H}}$, whilst the eigenrings will be computed on \widehat{H} . Also it is possible to apply the version of Kovacic's algorithm given in reference [62] to the algebrized operator \widehat{H} . The following results are obtained by applying Kovacic's algorithm to the reduced algebrized Schrödinger equation (equation (6.9)) $\widehat{\mathbf{H}}\Phi = \lambda\Phi$.

Proposition 6.17. *Let $\widehat{\mathbf{L}}_\lambda$ be the Picard-Vessiot extension of the reduced algebrized Schrödinger equation $\widehat{\mathbf{H}}\Phi = \lambda\Phi$ with $\alpha, \widehat{V} \in \mathbb{C}[z]$. If $\deg \alpha < 2 + \deg \widehat{V}$ then, for every $\lambda \in \Lambda$, $\text{DGal}(\widehat{\mathbf{L}}/\widehat{K})$ is not finite primitive.*

PROOF. Suppose that $\deg \alpha = n$ and $\deg \widehat{V} = m$. The reduced algebrized Schrödinger equation $\widehat{\mathbf{H}}\Phi = \lambda\Phi$ can be written in the form

$$\partial_z^2 \Phi = r\Phi, \quad r = \frac{4\alpha\partial_z^2 \alpha - 3(\partial_z \alpha)^2 + 16\alpha(\widehat{V} - \lambda)}{16\alpha^2}.$$

As $m > n - 2$ we have $\circ(r_\infty) = n - m < 2$, which does not satisfy the condition (∞) of case 3 of Kovacic's algorithm, therefore $\text{DGal}(\widehat{\mathbf{L}}/\widehat{K})$ is not finite primitive for any $\lambda \in \Lambda$.

□

Proposition 6.18. *Let $\widehat{\mathbf{L}}_\lambda$ be the Picard-Vessiot extension of the reduced algebrized Schrödinger equation $\widehat{\mathbf{H}}\Phi = \lambda\Phi$ with $\alpha \in \mathbb{C}[z]$, $\widehat{V} \in \mathbb{C}(z)$. If $\circ(\widehat{V})_\infty < 2 - \deg \alpha$ then, for every $\lambda \in \Lambda$, $\text{DGal}(\widehat{\mathbf{L}}_\lambda/\widehat{K})$ is not finite primitive.*

PROOF. Suppose that $\widehat{V} = s/t$, where s and t are coprime polynomials in $\mathbb{C}(z)$. Assume that $\deg \alpha = n$, $\deg s = m$ and $\deg t = p$. The reduced algebrized Schrödinger equation $\widehat{\mathbf{H}}\Phi = \lambda\Phi$ can be written in the form

$$\partial_z^2 \Phi = r\Phi, \quad r = \frac{4t\alpha\partial_z^2 \alpha - 3t(\partial_z \alpha)^2 + 16\alpha(s - \lambda t)}{16t\alpha^2}.$$

As $m > n + p - 2$, we have $\circ(r_\infty) = p + n - m < 2$, which does not satisfy condition (∞) of case 3 of Kovacic's algorithm. Therefore $\text{DGal}(\widehat{\mathbf{L}}/\widehat{K})$ is not finite primitive for any $\lambda \in \Lambda$.

□

Remark 6.19. In a natural way, we obtain the algebrized versions of Darboux transformation, i.e., the *algebrized Darboux transformation*, denoted by $\widehat{\text{DT}}$. The notation $\widehat{\text{DT}}_n$ stands for the n -th iteration of $\widehat{\text{DT}}$; the notation $\widehat{\text{CI}}_n$ denotes the n -th iteration of the *algebrized Crum iteration*, where the *algebrized wronskian* is given by

$$\widehat{W}(\widehat{y}_1, \dots, \widehat{y}_n) = \begin{vmatrix} \widehat{y}_1 & \cdots & \widehat{y}_n \\ \vdots & & \vdots \\ \widehat{\partial}_z^{n-1} \widehat{y}_1 & \cdots & \widehat{\partial}_z^{n-1} \widehat{y}_n \end{vmatrix}.$$

In the same way, we define *algebrized shape invariant potentials*, *algebrized superpotential* \widehat{W} , *algebrized supersymmetric Hamiltonians* \widehat{H}_\pm , *algebrized supersymmetric partner potentials* \widehat{V}_\pm , *algebrized ground state* $\widehat{\Psi}_0^{(-)} = e^{-\int \frac{\widehat{W}}{\widehat{V}_\pm} dz}$, *algebrized wave functions* $\widehat{\Psi}_\lambda^{(-)}$, *algebrized raising and lowering operators* \widehat{A} and \widehat{A}^\dagger . Thus, we can rewrite entirely section 5 using the derivation $\widehat{\partial}_z$.

The following theorem shows us the relationship between the algebrization and the Darboux transformation.

Theorem 6.20. *Given the Schrödinger equation $\mathcal{L}_\lambda := H_- \Psi^{(-)} = \lambda \Psi^{(-)}$, the relationship between the algebrization φ and Darboux transformations DT , $\widehat{\text{DT}}$ with respect to \mathcal{L}_λ is given by $\widehat{\text{DT}}\varphi = \varphi\text{DT}$, that is $\widehat{\text{DT}}\varphi(\mathcal{L}) = \varphi\text{DT}(\mathcal{L})$. In other words, the Darboux transformations DT and $\widehat{\text{DT}}$ are intertwined by the algebrization φ .*

PROOF. Consider the equations $\mathcal{L}_\lambda := H_- \Psi^{(-)} = \lambda \Psi^{(-)}$, $\widehat{\mathcal{L}}_\lambda := \widehat{H}_- \widehat{\Psi}^{(-)} = \lambda \widehat{\Psi}^{(-)}$, $\widetilde{\mathcal{L}}_\lambda := H_+ \Psi^{(+)} = \lambda \Psi^{(+)}$ and $\widetilde{\widehat{\mathcal{L}}}_\lambda := \widehat{H}_+ \widehat{\Psi}^{(+)} = \lambda \widehat{\Psi}^{(+)}$, where the Darboux transformations DT and $\widehat{\text{DT}}$ are given by $\text{DT}(\mathcal{L}) = \widetilde{\mathcal{L}}$, $\widehat{\text{DT}}(\widehat{\mathcal{L}}) = \widetilde{\widehat{\mathcal{L}}}$,

$$\text{DT} : \begin{array}{l} V_- \mapsto V_+ \\ \Psi_\lambda^{(-)} \mapsto \Psi_\lambda^{(+)} \end{array}, \quad \widehat{\text{DT}} : \begin{array}{l} \widehat{V}_- \mapsto \widehat{V}_+ \\ \widehat{\Psi}_\lambda^{(-)} \mapsto \widehat{\Psi}_\lambda^{(+)} \end{array},$$

and $\varphi(\mathcal{L}_\lambda) = \widehat{\mathcal{L}}_\lambda$, where the algebrization φ is given as in theorem 6.13. Then the following diagram commutes

$$\begin{array}{ccc} \mathcal{L}_\lambda & \xrightarrow{\text{DT}} & \widetilde{\mathcal{L}}_\lambda \\ \varphi \downarrow & & \downarrow \varphi \\ \widehat{\mathcal{L}}_\lambda & \xrightarrow{\widehat{\text{DT}}} & \widetilde{\widehat{\mathcal{L}}}_\lambda \end{array} \quad \text{so } \widehat{\text{DT}}\varphi(\mathcal{L}) = \varphi\text{DT}(\mathcal{L}) \text{ i.e. } \widetilde{\widehat{\mathcal{L}}} = \widehat{\widetilde{\mathcal{L}}}.$$

□

To illustrate $\widehat{\text{DT}}$ we present the following examples.

Examples. Consider the algebrized Schrödinger equation $\widehat{H}\widehat{\Psi}^{(-)} = \lambda\widehat{\Psi}^{(-)}$ with:

- $\sqrt{\alpha(z)} = \sqrt{z^2 - 1}$ and $\widehat{V}_-(z) = \frac{z}{z-1}$. Taking $\lambda_1 = 1$ and $\widehat{\Psi}_1^{(-)} = \sqrt{\frac{z+1}{z-1}}$, we have $\widehat{\text{DT}}(\widehat{V}_-) = \widehat{V}_+(z) = \frac{z}{z+1}$ and

$$\widehat{\text{DT}}(\widehat{\Psi}_\lambda^{(-)}) = \widehat{\Psi}_\lambda^{(+)} = \sqrt{z^2 - 1}\partial_z \widehat{\Psi}_\lambda^{(-)} + \frac{1}{\sqrt{z^2 - 1}}\widehat{\Psi}_\lambda^{(-)},$$

where $\widehat{\Psi}_\lambda^{(-)}$ is the general solution of $\widehat{H}_-\widehat{\Psi}^{(-)} = \lambda\widehat{\Psi}^{(-)}$ for $\lambda \neq 1$.

The original potential corresponding to this example is given by $V_-(x) = \frac{\cosh x}{\cosh x - 1}$ and for $\lambda_1 = 1$ the particular solution $\Psi_1^{(-)}$ is given by $\frac{\sinh x}{\cosh x - 1}$. Applying DT we have $\text{DT}(V_-) = V_+(x) = \frac{\cosh x}{\cosh x + 1}$ and $\text{DT}(\Psi_\lambda^{(-)}) = \Psi_\lambda^{(+)} = \partial_x \Psi_\lambda^{(-)} + \frac{1}{\sinh x}\Psi_\lambda^{(-)}$.

- $\sqrt{\alpha} = -z$, $\widehat{V}_-(z) = z^2 - z$. Taking $\lambda_1 = 0$ and $\widehat{\Psi}_0^{(-)} = e^{-z}$ we have $\widehat{\text{DT}}(\widehat{V}_-) = V_+ = z^2 + z$ and $\widehat{\text{DT}}(\widehat{\Psi}_\lambda^{(-)}) = \widehat{\Psi}_\lambda^{(+)} = -z\partial_z \widehat{\Psi}_\lambda^{(-)} - z\widehat{\Psi}_\lambda^{(-)}$, where $\Psi_\lambda^{(-)}$ is the general solution of $\widehat{H}_-\widehat{\Psi}^{(-)} = \lambda\widehat{\Psi}^{(-)}$ for $\lambda \neq 0$. This example corresponds to the *Morse potential* $V_-(x) = e^{-2x} - e^{-x}$.

To illustrate the algebrized Crum iteration $\widehat{\text{CI}}_n$, we present the following example, which is related with the Chebyshev polynomials.

Example. Consider $\sqrt{\alpha} = -\sqrt{1 - z^2}$, $V = 0$ with eigenvalues and eigenfunctions $\lambda_1 = 1$, $\lambda_2 = 4$, $\widehat{\Psi}_1 = z$, $\widehat{\Psi}_4 = 2z^2 - 1$, $\widehat{\Psi}_{n^2} = T_n(z)$, where $T_n(z)$ is the Chebyshev polynomial of first kind of degree n . The algebrized Wronskian for $n = 2$ is

$$\widehat{W}(z, 2z^2 - 1) = -\sqrt{1 - z^2}(2z^2 + 1),$$

and by algebrized Crum iteration we obtain the potential

$$\widehat{\text{CI}}_2(\widehat{V}) = \widehat{V}_2 = ((2z^2 - 1)\partial_z^2 + z\partial_z) \ln \widehat{W}(z, 2z^2 - 1)$$

and the algebrized wave functions

$$\widehat{\text{CI}}_2(\widehat{\Psi}_\lambda) = \widehat{\Psi}_\lambda^{(2)} = \frac{\widehat{W}(z, 2z^2 - 1, T_n)}{\widehat{W}(z, 2z^2 - 1)}.$$

In a natural way, we introduce the notion of algebrized shape invariant potentials $\widehat{V}_{n+1}(z, a_n) = \widehat{V}_n(z, a_{n+1}) + R(a_n)$, where the energy levels for $n > 0$ are given by $E_n = R(a_1) + \cdots + R(a_n)$ and the algebrized eigenfunctions are given by $\widehat{\Psi}_n(a_1) = \widehat{A}^\dagger(z, a_1) \cdots \widehat{A}^\dagger(z, a_n) \widehat{\Psi}_0(z, a_n)$. To illustrate the algebrized shape invariant potentials and the operators \widehat{A} and \widehat{A}^\dagger , we present the following example.

Example. Let $\sqrt{\alpha} = 1 - z^2$ and consider the algebrized super potential $\widehat{W}(z) = z$. Following the method proposed in remark 4.10, step 1, we introduce $\mu \in \mathbb{C}$ to obtain $\widehat{W}(z; \mu) = \mu z$, and

$$\widehat{V}_\pm(z; \mu) = \widehat{W}^2(z; \mu) \pm \partial_z \widehat{W}(z; \mu) = \mu(\mu \mp 1)z^2 \pm \mu.$$

Thus $\widehat{V}_+(z; a_0) = a_0(a_0 - 1)z^2 + a_0$ and $\widehat{V}_-(z; a_1) = a_1(a_1 + 1)z^2 - a_1$. By step 2,

$$\partial_z(V_+(z; a_0) - V_-(z; a_1)) = 2z(1 - z^2)(a_0(a_0 - 1) - a_1(a_1 + 1)).$$

By step 3, we obtain

$$a_1(a_1 + 1) = a_0(a_0 - 1), \quad a_1^2 - a_0^2 = -(a_1 + a_0)$$

and, assuming $a_1 \neq \pm a_0$, we have $a_1 = f(a_0) = a_0 - 1$ and $R(a_1) = 2a_0 + 1 = (a_0 + 1)^2 - a_0^2 = a_1^2 - a_0^2$. This means that the potentials \widehat{V}_\pm are algebrized shape invariant potentials where the n -th energy level $E = E_n$ is easily obtained:

$$E_n = \sum_{k=1}^n R(a_k) = \sum_{k=1}^n (a_k^2 - a_{k-1}^2) = a_n^2 - a_0^2, \quad a_n = f^n(a_0) = a_0 + n.$$

Now, the algebrized ground state wave function of $\widehat{V}_-(z, a_0)$ is

$$\widehat{\Psi}_0 = e^{\int \frac{a_0 z}{1-z^2} dz} = \frac{1}{(\sqrt{1-z^2})^{a_0}}.$$

Finally, we can obtain the rest of eigenfunctions using the algebrized raising operator:

$$\widehat{\Psi}_n(z, a_0) = \widehat{A}^\dagger(z, a_0) \widehat{A}^\dagger(z, a_1) \cdots \widehat{A}^\dagger(z, a_{n-1}) \widehat{\Psi}_0(z, a_n).$$

This example corresponds to the *Pöschl-Teller potential*.

Now to illustrate the power of Kovacic's algorithm with the derivation $\widehat{\partial}_z$, we study some Schrödinger equations for non-rational shape invariant potentials.

Morse potential: $V(x) = e^{-2x} - e^{-x}$.

The Schrödinger equation $H\Psi = \lambda\Psi$ is

$$\partial_x^2 \Psi = (e^{-2x} - e^{-x} - \lambda) \Psi.$$

By the Hamiltonian change of variable $z = z(x) = e^{-x}$, we obtain

$$\alpha(z) = z^2, \quad \widehat{V}(z) = z^2 - z, \quad \widehat{\mathbf{V}}(z) = \frac{z^2 - z - \frac{1}{4}}{z^2}.$$

Thus, $\widehat{K} = \mathbb{C}(z)$ and $K = \mathbb{C}(e^x)$. The algebrized Schrödinger equation $\widehat{H}\widehat{\Psi} = \lambda\widehat{\Psi}$ is

$$z^2 \partial_z^2 \widehat{\Psi} + z \partial_z \widehat{\Psi} - (z^2 - z - \lambda) \widehat{\Psi} = 0$$

and the reduced algebrized Schrödinger equation $\widehat{\mathbf{H}}\widehat{\Phi} = \lambda\widehat{\Phi}$ is

$$\partial_z^2 \Phi = r\Phi, \quad r = \frac{z^2 - z - \frac{1}{4} - \lambda}{z^2}.$$

This equation could only fall under case 1, in case 2 or in case 4 (of Kovacic's algorithm). We start by analyzing the case 1: by conditions c_2 and ∞_3 we have that

$$[\sqrt{r}]_0 = 0, \quad \alpha_0^\pm = \frac{1 \pm 2\sqrt{-\lambda}}{2}, \quad [\sqrt{r}]_\infty = 1, \quad \text{and} \quad \alpha_\infty^\pm = \mp \frac{1}{2}.$$

By step 2 we have the following possibilities for $n \in \mathbb{Z}_+$ and for $\lambda \in \Lambda$:

$$\Lambda_{++}) \quad n = \alpha_\infty^+ - \alpha_0^+ = -1 - \sqrt{-\lambda}, \quad \lambda = -(n+1)^2,$$

$$\Lambda_{+-}) \quad n = \alpha_\infty^+ - \alpha_0^- = -1 + \sqrt{-\lambda}, \quad \lambda = -(n+1)^2,$$

$$\Lambda_{-+}) \quad n = \alpha_\infty^- - \alpha_0^+ = -\sqrt{-\lambda}, \quad \lambda = -n^2,$$

$$\Lambda_{--}) \quad n = \alpha_\infty^- - \alpha_0^- = \sqrt{-\lambda}, \quad \lambda = -n^2.$$

We can see that $\lambda \in \Lambda_- = \{-n^2 : n \in \mathbb{Z}_+\}$. Now, for $\lambda \in \Lambda$, the rational function ω is given by:

$$\Lambda_{++}) \quad \omega = 1 + \frac{3+2n}{2z}, \quad \lambda \in \Lambda_{++}, \quad r_n = \frac{4n^2+8n+3}{4z^2} + \frac{2n+3}{z} + 1,$$

$$\Lambda_{+-}) \quad \omega = 1 - \frac{1+2n}{2z}, \quad \lambda \in \Lambda_{+-}, \quad r_n = \frac{4n^2+8n+3}{4z^2} - \frac{2n+1}{z} + 1,$$

$$\Lambda_{-+}) \quad \omega = -1 + \frac{1+2n}{2z}, \quad \lambda \in \Lambda_{-+}, \quad r_n = \frac{4n^2-1}{4z^2} - \frac{2n+1}{z} + 1,$$

$$\Lambda_{--}) \quad \omega = -1 + \frac{1-2n}{2z}, \quad \lambda \in \Lambda_{--}, \quad r_n = \frac{4n^2-1}{4z^2} + \frac{2n-1}{z} + 1,$$

where r_n is the coefficient of the differential equation $\partial_z^2 \Phi = r_n \Phi$.

By step 3, there exists a polynomial of degree n satisfying the relation (A.1),

$$\Lambda_{++}) \quad \partial_z^2 \widehat{P}_n + 2 \left(1 + \frac{3+2n}{2z}\right) \partial_z \widehat{P}_n + \frac{2(n+2)}{z} \widehat{P}_n = 0,$$

$$\Lambda_{+-}) \quad \partial_z^2 \widehat{P}_n + 2 \left(1 - \frac{1+2n}{2z}\right) \partial_z \widehat{P}_n + \frac{2(-n)}{z} \widehat{P}_n = 0,$$

$$\Lambda_{-+}) \quad \partial_z^2 \widehat{P}_n + 2 \left(-1 + \frac{1+2n}{2z}\right) \partial_z \widehat{P}_n + \frac{2(-n)}{z} \widehat{P}_n = 0,$$

$$\Lambda_{--}) \quad \partial_z^2 \widehat{P}_n + 2 \left(-1 + \frac{1-2n}{2z}\right) \partial_z \widehat{P}_n + \frac{2n}{z} \widehat{P}_n = 0.$$

These polynomials only exist for $n = \lambda = 0$, with $\lambda \in \Lambda_{-+} \cup \Lambda_{--}$. So the solutions of $H\Psi = 0$, $\widehat{H}\widehat{\Psi} = 0$ and $\widehat{\mathbf{H}}\widehat{\Phi} = 0$ are given by

$$\Phi_0 = \sqrt{z}e^{-z}, \quad \widehat{\Psi}_0 = e^{-z}, \quad \Psi = e^{-e^{-x}}.$$

The wave function Ψ_0 satisfy the bound state conditions, which means that is ground state (see [23]) and $0 \in \text{Spec}_p(H)$. Furthermore, we have

$$\text{DGal}(L_0/K) = \text{DGal}(\widehat{L}_0/\widehat{K}) = \text{DGal}(\widehat{\mathbf{L}}_0/\mathbb{C}(z)) = \mathbb{B},$$

$$\mathcal{E}(H) = \mathcal{E}(\widehat{H}) = \mathcal{E}(\widehat{\mathbf{H}}) = \text{Vect}(1).$$

We proceed with case 2. The conditions c_2 and ∞_3 are satisfied. So we have

$$E_c = \left\{ 2, 2 + 4\sqrt{-\lambda}, 2 - 4\sqrt{-\lambda} \right\} \quad \text{and} \quad E_\infty = \{0\},$$

and by step two, we have that $2 \pm \sqrt{-\lambda} = m \in \mathbb{Z}_+$, so that $\lambda = -\left(\frac{m+1}{2}\right)^2$ and the rational function θ has the following possibilities

$$\theta_+ = \frac{2+m}{z}, \quad \theta_- = -\frac{m}{z}.$$

By step three, there exist a monic polynomial of degree m satisfying the recurrence relation (A.2):

$$\theta_+ \quad \partial_z^3 \widehat{P}_m + \frac{3m+6}{z} \partial_z^2 \widehat{P}_m - \frac{4z^2-4z-2m^2-7m-6}{z^2} \partial_z \widehat{P}_m - \frac{4mz+8z-4m-6}{z^2} \widehat{P}_m = 0,$$

$$\theta_- \quad \partial_z^3 \widehat{P}_m - \frac{3m}{z} \partial_z^2 \widehat{P}_m - \frac{4z^2-4z-2m^2-m}{z^2} \partial_z \widehat{P}_m + \frac{4mz-4m-2}{z^2} \widehat{P}_m = 0.$$

We can see that, for $m = 1$, the polynomial exists only in the case θ_- , with $\widehat{P}_1 = z - 1/2$. In general, these polynomials could exist only in the case θ_- with $m = 2n - 1$, $n \geq 1$, that is $\lambda \in \{-n^2 : n \geq 1\}$.

By case one and case two, we obtain $\Lambda = \{-n^2 : n \geq 0\} = \text{Spec}_p(H)$. Now, the rational function ϕ and the quadratic expression for ω are

$$\phi = -\frac{2n-1}{z} + \frac{\partial_z \widehat{P}_{2n-1}}{\widehat{P}_{2n-1}}, \quad \omega^2 + M\omega + N = 0, \quad \omega = \frac{-M \pm \sqrt{M^2 - 4N}}{2},$$

where the coefficients M and N are given by

$$M = \frac{2n-1}{z} - \frac{\partial_z \widehat{P}_{2n-1}}{\widehat{P}_{2n-1}}, \quad N = \frac{n^2 - n + \frac{1}{4}}{z^2} - \frac{(2n-1) \frac{\partial_z \widehat{P}_{2n-1}}{\widehat{P}_{2n-1}} - 2}{z} + \frac{\partial_z^2 \widehat{P}_{2n-1}}{\widehat{P}_{2n-1}} - 2.$$

Now, $\Delta = M^2 - 4N \neq 0$, which means that $\widehat{\mathbf{H}}\Phi = -n^2\Phi$ with $n \in \mathbb{Z}^+$ has two solutions given by Kovacic's algorithm:

$$\Phi_{1,n} = \frac{\sqrt{z} \widehat{P}_n e^{-z}}{z^n}, \quad \Phi_{2,n} = \frac{\sqrt{z} \widehat{P}_{n-1} e^z}{z^n}.$$

The solutions of $\widehat{H}\widehat{\Psi} = -n^2\widehat{\Psi}$ are given by

$$\widehat{\Psi}_{1,n} = \frac{\widehat{P}_n e^{-z}}{z^n}, \quad \widehat{\Psi}_{2,n} = \frac{\widehat{P}_{n-1} e^z}{z^n},$$

and therefore, the solutions of the Schrödinger equation $H\Psi = -n^2\Psi$ are

$$\Psi_{1,n} = P_n e^{-e^{-x}} e^{nx}, \quad \Psi_{2,n} = P_{n-1} e^{e^{-x}} e^{nx}, \quad P_n = \widehat{P}_n \circ z.$$

The wave functions $\Psi_{1,n} = \Psi_n$ satisfy the conditions of bound state and, for $n = 0$, this solution coincides with the ground state presented above. Therefore we have

$$\Phi_n = \Phi_0 \widehat{f}_n \widehat{P}_n, \quad \widehat{\Psi}_n = \widehat{\Psi}_0 \widehat{f}_n \widehat{P}_n, \quad \widehat{f}_n(z) = \frac{1}{z^n}.$$

Thus, the bound states wave functions are obtained as

$$\Psi_n = \Psi_0 f_n P_n, \quad f_n(x) = \widehat{f}_n(e^{-x}) = e^{nx}.$$

The eigenring s and differential Galois groups for $n > 0$ satisfies

$$\begin{aligned} \mathrm{DGal}(L_n/K) &= \mathrm{DGal}(\widehat{L}_n/\widehat{K}) = \mathrm{DGal}(\widehat{\mathbf{L}}_n/\mathbb{C}(z)) = \mathbb{G}_m, \\ \dim_{\mathbb{C}} \mathcal{E}(\widehat{\mathbf{H}} + n^2) &= \dim_{\mathbb{C}} \mathcal{E}(\widehat{H} + n^2) = \dim_{\mathbb{C}} \mathcal{E}(H + n^2) = 2. \end{aligned}$$

We remark that the Schrödinger equation with Morse potential, under suitable changes of variables [38], may be transformed to a Bessel differential equation.

It is known that equations deriving from Eckart, Rosen-Morse, Scarf and Pöschl-Teller potentials may be mapped, under suitable transformations, to Hypergeometric equations. These potentials are inter-related by point canonical coordinate transformations (see [16, p. 314]), so that $\Lambda = \mathbb{C}$ because Pöschl-Teller potential is obtained by means of Darboux transformations of $V = 0$ ([42, 52]). We consider some particular cases of Eckart, Scarf and Poschl-Teller potentials for which we apply only case 1 of Kovacic's algorithm. Case 1 allows us to obtain the enumerable set $\Lambda_n \subset \Lambda$, which includes the classical results obtained by means of supersymmetric quantum mechanics. Cases 2 and 3 of Kovacic algorithm also can be applied, but are not considered here.

Eckart potential: $V(x) = 4 \coth(x) + 5$, $x > 0$.

The Schrödinger equation $H\Psi = \lambda\Psi$ is

$$\partial_x^2 \Psi = (4 \coth(x) + 5 - \lambda) \Psi.$$

By the Hamiltonian change of variable $z = z(x) = \coth(x)$, we obtain

$$\alpha(z) = (1 - z^2)^2, \quad \widehat{V}(z) = 4z + 5, \quad \widehat{\mathbf{V}}(z) = \frac{4}{(z+1)(z-1)^2}.$$

Thus, $\widehat{K} = \mathbb{C}(z)$ and $K = \mathbb{C}(\coth(x))$. The algebrized Schrödinger equation $\widehat{H}\widehat{\Psi} = \lambda\widehat{\Psi}$ is

$$(1 - z^2)^2 \partial_z^2 \widehat{\Psi} - 2z(1 - z^2) \partial_z \widehat{\Psi} - (4z + 5 - \lambda) \widehat{\Psi} = 0$$

and the reduced algebrized Schrödinger equation $\widehat{\mathbf{H}}\widehat{\Phi} = \lambda\widehat{\Phi}$ is

$$\partial_z^2 \Phi = r\Phi, \quad r = \frac{4z + 4 - \lambda}{(z-1)^2(z+1)^2} = \frac{2 - \frac{\lambda}{4}}{(z-1)^2} + \frac{\frac{\lambda}{4} - 1}{(z-1)} + \frac{-\frac{\lambda}{4}}{(z+1)^2} + \frac{1 - \frac{\lambda}{4}}{(z+1)}$$

We can see that this equation could fall in any case of Kovacic's algorithm. Considering $\lambda = 0$, the conditions $\{c_1, c_2, \infty_1\}$ of case 1 are satisfied, obtaining

$$[\sqrt{r}]_{-1} = [\sqrt{r}]_1 = [\sqrt{r}]_{\infty} = \alpha_{\infty}^+ = 0, \quad \alpha_{-1}^{\pm} = \alpha_{\infty}^- = 1, \quad \alpha_1^+ = 2, \quad \alpha_1^- = -1.$$

By step 2, the elements of D are 0 and 1. The rational function ω for $n = 0$ and for $n = 1$ must be

$$\omega = \frac{1}{z+1} + \frac{-1}{z-1}.$$

By step 3, we search for a monic polynomial of degree n satisfying the relation (A.1). Starting with $n = 0$ the only one possibility is $\widehat{P}_0(z) = 1$, which effectively satisfy the relation (A.1), while $\widehat{P}_1(z) = z + a_0$ does not exist. We have obtained one solution using Kovacic algorithm:

$$\Phi_0 = \frac{z+1}{z-1}, \quad \widehat{\Psi}_0 = \sqrt{\frac{z+1}{(z-1)^3}},$$

this means that $0 \in \Lambda_n$. We can obtain the second solution using the first solution:

$$\Phi_{0,2} = \frac{z^2 + z - 4 - 4 \ln(z+1)z - 4 \ln(z+1)}{z-1}, \quad \widehat{\Psi}_{0,2} = \frac{\Phi_{0,2}}{\sqrt{z^2-1}}.$$

Furthermore the differential Galois groups and eigenring s for $\lambda = 0$ are

$$\mathrm{DGal}(\widehat{\mathbf{L}}_0/\mathbb{C}(z)) = \mathbb{G}_a, \quad \mathrm{DGal}(L_0/K) = \mathrm{DGal}(\widehat{L}_0/\widehat{K}) = \mathbb{G}^{\{2\}},$$

$$\mathcal{E}(\widehat{\mathbf{H}}) = \mathrm{Vect} \left(1, \frac{(z+1)^2}{(1-z)^2} \partial_z + \frac{2(z+1)}{(1-z)^3} \right),$$

$$\mathcal{E}(\widehat{H}) = \mathrm{Vect} \left(1, \frac{(z+1)^2}{(1-z)^2} \partial_z - \frac{z^2+3z+2}{(1-z)^3} \right),$$

$$\mathcal{E}(H) = \mathrm{Vect} \left(1, \frac{(\coth(x)+1)^2}{(1-\coth(x))^2(1-\coth^2(x))} \partial_x - \frac{\coth^2(x)+3\coth(x)+2}{(1-\coth(x))^3} \right).$$

Now, for $\lambda \neq 0$, the conditions $\{c_2, \infty_1\}$ of case 1 are satisfied:

$$[\sqrt{r}]_{-1} = [\sqrt{r}]_1 = [\sqrt{r}]_\infty = \alpha_\infty^+ = 0, \quad \alpha_\infty^- = 1,$$

$$\alpha_{-1}^\pm = \frac{1 \pm \sqrt{1-\lambda}}{2}, \quad \alpha_1^\pm = \frac{1 \pm \sqrt{9-\lambda}}{2}.$$

By step 2 we have the following possibilities for $n \in \mathbb{Z}_+$ and for $\lambda \in \Lambda$:

$$\Lambda_{++-}) \quad n = \alpha_\infty^+ - \alpha_{-1}^+ - \alpha_1^- = -1 - \frac{\sqrt{1-\lambda}-\sqrt{9-\lambda}}{2}, \quad \lambda = 4 - \frac{4}{(n+1)^2} - n^2 - 2n,$$

$$\Lambda_{+--}) \quad n = \alpha_\infty^+ - \alpha_{-1}^- - \alpha_1^- = -1 + \frac{\sqrt{1-\lambda}+\sqrt{9-\lambda}}{2}, \quad \lambda = 4 - \frac{4}{(n+1)^2} - n^2 - 2n,$$

$$\Lambda_{-+-}) \quad n = \alpha_\infty^- - \alpha_{-1}^+ - \alpha_1^- = \frac{\sqrt{1-\lambda}+\sqrt{9-\lambda}}{2}, \quad \lambda = 5 - \frac{4}{n^2} - n^2,$$

$$\Lambda_{---}) \quad n = \alpha_\infty^- - \alpha_{-1}^- - \alpha_1^- = \frac{\sqrt{1-\lambda}+\sqrt{9-\lambda}}{2}, \quad \lambda = 5 - \frac{4}{n^2} - n^2.$$

Therefore, we have

$$\Lambda_n \subseteq \left\{ 4 - \frac{4}{(n+1)^2} - n^2 - 2n : n \in \mathbb{Z}_+ \right\} \cup \left\{ 5 - \frac{4}{n^2} - n^2 : n \in \mathbb{Z}_+ \right\}.$$

Now, for $\lambda \in \Lambda$, the rational function ω is given by:

$$\Lambda_{++-}) \quad \omega = \frac{z(n-1)-n^2-2n-1}{(n+1)(z+1)(z-1)}, \quad r_n = \frac{-2z^2(n-1)+4z(n+1)^2+(n+1)(n^3+3n^2+2n+2)}{(n+1)^2(z+1)^2(z-1)^2},$$

$$\Lambda_{+--}) \quad \omega = \frac{nz(n+1)+2}{(n+1)(z+1)(1-z)}, \quad r_n = \frac{nz^2(n+1)^3+4z(n+1)^2+n^3+2n^2+n+4}{(n+1)^2(z+1)^2(z-1)^2},$$

$$\Lambda_{-+-}) \quad \omega = \frac{z(n-2)-n^2}{n(z+1)(z-1)}, \quad r_n = \frac{-2z^2(n-2)+4n^2z+n(n^3-n+2)}{n^2(z+1)^2(z-1)^2},$$

$$\Lambda_{---}) \quad \omega = \frac{nz(n-1)+2}{n(z+1)(1-z)}, \quad r_n = \frac{n^3z^2(n-1)+4n^2z+n^3-n^2+4}{n^2(z+1)^2(z-1)^2},$$

where r_n is the coefficient of the differential equation $\partial_z^2 \Phi = r_n \Phi$.

By step 3, there exists a monic polynomial of degree n satisfying the relation (A.1),

$$\begin{aligned}
\Lambda_{++-}) \quad & \partial_z^2 \hat{P}_n + 2 \left(\frac{z(n-1)-n^2-2n-1}{(n+1)(z+1)(z-1)} \right) \partial_z \hat{P}_n + \frac{2(1-n)}{(n+1)^2(z+1)(z-1)} \hat{P}_n = 0, \\
\Lambda_{+--}) \quad & \partial_z^2 \hat{P}_n + 2 \left(\frac{nz(n+1)+2}{(n+1)(z+1)(1-z)} \right) \partial_z \hat{P}_n + \frac{n(n+1)}{(z+1)(z-1)} \hat{P}_n = 0, \\
\Lambda_{-+-}) \quad & \partial_z^2 \hat{P}_n + 2 \left(\frac{z(n-2)-n^2}{n(z+1)(z-1)} \right) \partial_z \hat{P}_n + \frac{2(2-n)}{n^2(z+1)(z-1)} \hat{P}_n = 0, \\
\Lambda_{---}) \quad & \partial_z^2 \hat{P}_n + 2 \left(\frac{nz(n-1)+2}{n(z+1)(1-z)} \right) \partial_z \hat{P}_n + \frac{n(n-1)}{(z+1)(z-1)} \hat{P}_n = 0.
\end{aligned}$$

The only one case in which there exists such a polynomial \hat{P}_n of degree n is for $\Lambda_{+--})$. The solutions of the equation $\hat{\mathbf{H}}\Phi = \lambda\Phi$, with $\lambda \neq 0$, are:

$$\begin{aligned}
\Lambda_{++-}) \quad & \Phi_n = \hat{P}_n \hat{f}_n \Phi_0, \quad \Phi_0 = \frac{1}{z-1} \quad \hat{f}_n = (z-1)^{\frac{n(1-n)}{2(n+1)}} (z+1)^{\frac{n(n+3)}{2(n+1)}}, \\
\Lambda_{+--}) \quad & \Phi_n = \hat{P}_n \hat{f}_n \Phi_0, \quad \Phi_0 = \frac{z+1}{z-1} \quad \hat{f}_n = (z-1)^{\frac{n(1-n)}{2(n+1)}} (z+1)^{\frac{-n(n+3)}{2(n+1)}}, \\
\Lambda_{-+-}) \quad & \Phi_n = \hat{P}_n \hat{f}_n \Phi_1, \quad \Phi_1 = \frac{1}{z-1} \quad \hat{f}_n = (z+1)^{\frac{n^2+n-2}{2n}} (z-1)^{\frac{-n^2+3n-2}{2n}}, \\
\Lambda_{---}) \quad & \Phi_n = \hat{P}_n \hat{f}_n \Phi_1, \quad \Phi_1 = \frac{z+1}{z-1} \quad \hat{f}_n = (z+1)^{\frac{-n^2-n+2}{2n}} (z-1)^{\frac{-n^2+3n-2}{2n}}.
\end{aligned}$$

In any case $\hat{\Psi}_n = \frac{\Phi_n}{1-z^2}$. The case $\Lambda_{+--})$ includes the classical results obtained by means of supersymmetric quantum mechanics. Thus, replacing z by $\coth(x)$ we obtain the eigenstates Ψ_n . The eigenring s and differential Galois groups for $n > 0$ and $\lambda \in \Lambda_n$ satisfy

$$\mathrm{DGal}(L_\lambda/K) \subseteq \mathbb{G}^{\{2m\}}, \quad \mathrm{DGal}(\hat{L}_\lambda/\hat{K}) \subseteq \mathbb{G}^{\{2m\}},$$

$$\mathrm{DGal}(\hat{\mathbf{L}}_\lambda/\mathbb{C}(z)) = \mathbb{G}_m,$$

$$\dim_{\mathbb{C}(z)} \mathcal{E}(\hat{\mathbf{H}} + \lambda) = 2, \quad \mathcal{E}(\hat{H} + \lambda) = \mathcal{E}(H + \lambda) = \mathrm{Vect}(1).$$

Scarf potential: $V(x) = \frac{\sinh^2 x - 3 \sinh x}{\cosh^2 x}.$

The Schrödinger equation $H\Psi = E\Psi$ is

$$\partial_x^2 \Psi = \left(\frac{\sinh^2 x - 3 \sinh x}{\cosh^2 x} - E \right) \Psi.$$

By the Hamiltonian change of variable $z = z(x) = \sinh(x)$, we obtain

$$\alpha(z) = 1 + z^2, \quad \hat{V}(z) = \frac{z^2 - 3z}{1 + z^2}.$$

Thus, $\hat{K} = \mathbb{C}(z, \sqrt{1+z^2})$ and $K = \mathbb{C}(\sinh(x), \cosh(x))$. The reduced algebrized Schrödinger equation $\hat{\mathbf{H}}\Phi = \lambda\Phi$ is

$$\partial_z^2 \Phi = \left(\frac{\lambda z^2 - 12z + \lambda - 1}{4(z^2 + 1)^2} \right) \Phi, \quad \lambda = 3 - 4E.$$

Applying Kovacic's algorithm for this equation with $\lambda = 0$, we see that it does not satisfy case 1. So we consider only $\lambda \neq 0$. By conditions $\{c_2, \infty_2\}$ of case 1 we have

$$[\sqrt{r}]_{-i} = [\sqrt{r}]_i = [\sqrt{r}]_\infty = 0, \quad \alpha_\infty^\pm = \frac{1 \pm \sqrt{1+\lambda}}{2},$$

$$\alpha_{-i}^+ = \frac{5}{4} - \frac{i}{2}, \quad \alpha_{-i}^- = -\frac{1}{4} + \frac{i}{2}, \quad \alpha_i^+ = \frac{5}{4} + \frac{i}{2}, \quad \alpha_i^- = -\frac{1}{4} - \frac{i}{2}.$$

By step 2 we have the following possibilities for $n \in \mathbb{Z}_+$ and for $\lambda \in \Lambda$:

$$\Lambda_{+++}) \quad n = \alpha_\infty^+ - \alpha_{-i}^+ - \alpha_i^+ = \frac{\sqrt{\lambda+1}-4}{2}, \quad \lambda = 4n^2 + 16n + 15,$$

$$\Lambda_{+--}) \quad n = \alpha_\infty^+ - \alpha_{-i}^- - \alpha_i^- = \frac{\sqrt{\lambda+1}+2}{2}, \quad \lambda = 4n^2 - 8n + 3,$$

thus obtaining

$$\Lambda_n \subseteq \{4n^2 + 16n + 15 : n \in \mathbb{Z}_+\} \cup \{4n^2 - 8n + 3 : n \in \mathbb{Z}_+\}.$$

Now, the rational function ω is given by:

$$\Lambda_{+++}) \quad \omega = \frac{5z-2}{2(z^2+1)}, \quad \Lambda_{+--}) \quad \frac{2-z}{2(z^2+1)}.$$

By step 3, there exists $\hat{P}_0 = 1$ and a polynomial of degree $n \geq 1$ should satisfy either of the relation (A.1),

$$\Lambda_{+++}) \quad \partial_z^2 \hat{P}_n + \frac{5z-2}{z^2+1} \partial_z \hat{P}_n - \frac{nz^2(n+4)+3z+n^2+4n-3}{(z^2+1)^2} \hat{P}_n = 0,$$

$$\Lambda_{+--}) \quad \partial_z^2 \hat{P}_n + \frac{2-z}{z^2+1} \partial_z \hat{P}_n - \frac{nz^2(n-2)+3z+n^2-2n-3}{(z^2+1)^2} \hat{P}_n = 0.$$

In both cases there exists such a polynomial \hat{P}_n of degree $n \geq 1$. Basis of solutions $\{\Phi_{1,n}, \Phi_{2,n}\}$ of the reduced algebrized Schrödinger equation are:

$$\Lambda_{+++}) \quad \Phi_{1,n} = \hat{P}_n \hat{f}_n \Phi_{1,0}, \quad \Phi_{1,0} = (1+z^2)^{\frac{5}{4}} e^{-\arctan z}, \quad \hat{f}_n = 1,$$

$$\Phi_{2,n} = \hat{Q}_n \hat{g}_n \Phi_{2,0}, \quad \Phi_{2,0} = \frac{22+21x+12x^2+6x^3}{\sqrt{1+z^2}} e^{-\arctan z}, \quad \hat{g}_n = 1.$$

$$\Lambda_{+--}) \quad \Phi_{1,n} = \hat{P}_n \hat{f}_n \Phi_{1,0}, \quad \Phi_{1,0} = \frac{1}{\sqrt[4]{1+z^2}} e^{\arctan z}, \quad \hat{f}_n = 1,$$

$$\Phi_{2,n} = \hat{Q}_n \hat{g}_n \Phi_{2,0}, \quad \Phi_{2,0} = \frac{1}{\sqrt[4]{1+z^2}} e^{\arctan z} \int \frac{1}{\sqrt{1+z^2}} e^{-2\arctan z} dz, \quad \hat{g}_n = 1.$$

In both cases $\hat{\Psi} = \frac{\Phi}{\sqrt[4]{1+z^2}}$, but the classical case (see references [16, 23]) is $\Lambda_{+--})$, so that replacing z by $\sinh x$ and λ by $3 - 4E$ we obtain the eigenstates Ψ_n .

The eigenring s and differential Galois groups are

$$\mathcal{E}(H - \lambda) = \mathcal{E}(\hat{H} - \lambda) = \mathcal{E}(\hat{\mathbf{H}} - \lambda) = \text{Vect}(1),$$

$$\text{DGal}(L_\lambda/K) = \text{DGal}(\hat{L}_\lambda/\hat{K}) = \text{DGal}(\hat{\mathbf{L}}_\lambda/\mathbb{C}(x)) = \mathbb{B}.$$

Pöschl-Teller potential (revisited): $V(r) = \frac{\cosh^4(x) - \cosh^2(x) + 2}{\sinh^2(x) \cosh^2(x)}$, $x > 0$. The reduced algebrized Schrödinger equation $\hat{\mathbf{H}}\Phi = E\Phi$ is

$$\partial_z^2 \Phi = \left(\frac{\lambda z^4 - (\lambda + 3)z^2 + 8}{4z^2(z^2 - 1)^2} \right) \Phi, \quad \lambda = 3 - 4E.$$

Considering $\lambda = 0$ and starting with the conditions $\{c_2, \infty_1\}$ of case 1, we obtain

$$\begin{aligned} [\sqrt{r}]_0 &= [\sqrt{r}]_{-1} = [\sqrt{r}]_1 = [\sqrt{r}]_\infty = \alpha_\infty^+ = 0, & \alpha_\infty^- &= 1, \\ \alpha_{-1}^+ &= \alpha_1^+ = \frac{5}{4}, & \alpha_{-1}^- &= \alpha_1^- = -\frac{1}{4}, & \alpha_0^+ &= 2, & \alpha_0^- &= -1. \end{aligned}$$

By step 2, the elements of D are 0 and 1. For the rational function ω , we have the following possibilities for $n = 0$ and for $n = 1$:

$$\begin{aligned} \Lambda_{++--}) \quad n = 0, \quad \omega &= \frac{5/4}{z+1} + \frac{-1/4}{z-1} + \frac{-1}{z}, \\ \Lambda_{+-+-}) \quad n = 0, \quad \omega &= \frac{-1/4}{z+1} + \frac{5/4}{z-1} + \frac{-1}{z}, \\ \Lambda_{-+--}) \quad n = 1, \quad \omega &= \frac{5/4}{z+1} + \frac{-1/4}{z-1} + \frac{-1}{z}, \\ \Lambda_{--+-}) \quad n = 1, \quad \omega &= \frac{-1/4}{z+1} + \frac{5/4}{z-1} + \frac{-1}{z}. \end{aligned}$$

Following step 3, we search for a monic polynomial of degree n satisfying relation (A.1). Starting with $n = 0$ the only possibility for $\Lambda_{++--})$ and $\Lambda_{+-+-})$ is $\hat{P}_0(z) = 1$, which does not satisfy relation (A.1) in both cases, while $\hat{P}_1(z) = z + a_0$ effectively does exist, taking $a_0 = -\frac{2}{3}$ for $\Lambda_{-+--})$ and $a_0 = \frac{2}{3}$ for $\Lambda_{--+-})$. This way, we have obtained two solutions $(\Phi_{1,0}, \Phi_{2,0})$ using Kovacic's algorithm:

$$\begin{aligned} \Phi_{1,0} &= \left(1 - \frac{2}{3z}\right) \sqrt[4]{\frac{(z+1)^5}{z-1}}, & \hat{\Psi}_{1,0} &= \left(1 - \frac{2}{3z}\right) \frac{z+1}{\sqrt{z-1}}, \\ \Phi_{2,0} &= \left(1 + \frac{2}{3z}\right) \sqrt[4]{\frac{(z-1)^5}{z+1}}, & \hat{\Psi}_{2,0} &= \left(1 + \frac{2}{3z}\right) \frac{z-1}{\sqrt{z+1}}. \end{aligned}$$

This means that $0 \in \Lambda_n$. Furthermore,

$$\mathrm{DGal}(\hat{\mathbf{L}}_0/\mathbb{C}(x)) = \mathbb{G}^{[4]}, \quad \mathrm{DGal}(\hat{L}_0/\hat{K}) = \mathrm{DGal}(L_0/K) = e,$$

$$\dim_{\mathbb{C}} \mathcal{E}(\hat{\mathbf{H}}) = 2, \quad \dim_{\mathbb{C}} \mathcal{E}(\hat{H}) = \dim_{\mathbb{C}} \mathcal{E}(H) = 4.$$

Now, for $\lambda \neq 0$, we see that conditions $\{c_2, \infty_1\}$ of case 1 leads us to

$$\begin{aligned} [\sqrt{r}]_0 &= [\sqrt{r}]_{-1} = [\sqrt{r}]_1 = [\sqrt{r}]_\infty = 0, & \alpha_\infty^\pm &= \frac{1 \pm \sqrt{1+\lambda}}{2}, \\ \alpha_{-1}^+ &= \alpha_1^+ = \frac{5}{4}, & \alpha_{-1}^- &= \alpha_1^- = -\frac{1}{4}, & \alpha_0^+ &= 2, & \alpha_0^- &= -1. \end{aligned}$$

By step 2 we have the following possibilities for $n \in \mathbb{Z}_+$ and for $\lambda \in \Lambda$:

$$\begin{aligned}
 \Lambda_{++++}) \quad n &= \alpha_\infty^+ - \alpha_{-1}^+ - \alpha_1^+ - \alpha_0^+ = \frac{\sqrt{\lambda+1}-8}{2}, \quad \lambda = 4n^2 + 32n + 63, \\
 \Lambda_{+++ -}) \quad n &= \alpha_\infty^+ - \alpha_{-1}^+ - \alpha_1^+ - \alpha_0^- = \frac{\sqrt{\lambda+1}-2}{2}, \quad \lambda = 4n^2 + 8n + 3, \\
 \Lambda_{++ - +}) \quad n &= \alpha_\infty^+ - \alpha_{-1}^+ - \alpha_1^- - \alpha_0^+ = \frac{\sqrt{\lambda+1}-5}{2}, \quad \lambda = 4n^2 + 20n + 24, \\
 \Lambda_{++ - -}) \quad n &= \alpha_\infty^+ - \alpha_{-1}^+ - \alpha_1^- - \alpha_0^- = \frac{\sqrt{\lambda+1}+1}{2}, \quad \lambda = 4n^2 - 4n, \\
 \Lambda_{+ - ++}) \quad n &= \alpha_\infty^+ - \alpha_{-1}^- - \alpha_1^+ - \alpha_0^+ = \frac{\sqrt{\lambda+1}-5}{2}, \quad \lambda = 4n^2 + 20n + 24, \\
 \Lambda_{+ - +-}) \quad n &= \alpha_\infty^+ - \alpha_{-1}^- - \alpha_1^+ - \alpha_0^- = \frac{\sqrt{\lambda+1}+1}{2}, \quad \lambda = 4n^2 - 4n, \\
 \Lambda_{+ - - +}) \quad n &= \alpha_\infty^+ - \alpha_{-1}^- - \alpha_1^- - \alpha_0^+ = \frac{\sqrt{\lambda+1}-2}{2}, \quad \lambda = 4n^2 + 8n + 3, \\
 \Lambda_{+ - - -}) \quad n &= \alpha_\infty^+ - \alpha_{-1}^- - \alpha_1^- - \alpha_0^- = \frac{\sqrt{\lambda+1}+4}{2}, \quad \lambda = 4n^2 - 16n + 15,
 \end{aligned}$$

obtaining $\Lambda_n \subseteq \Lambda_a \cup \Lambda_b \cup \Lambda_c \cup \Lambda_d \cup \Lambda_e$, where

$$\begin{aligned}
 \Lambda_a &= \{4n^2 + 32n + 63 : n \in \mathbb{Z}_+\}, \quad \Lambda_b = \{4n^2 + 8n + 3 : n \in \mathbb{Z}_+\} \\
 \Lambda_c &= \{4n^2 + 20n + 24 : n \in \mathbb{Z}_+\}, \quad \Lambda_d = \{4n^2 - 4n : n \in \mathbb{Z}_+\}, \\
 \Lambda_e &= \{4n^2 - 16n + 15 : n \in \mathbb{Z}_+\}.
 \end{aligned}$$

Now, the rational function ω is given by, respectively:

$$\begin{aligned}
 \Lambda_{++++}) \quad \omega &= \frac{5/4}{z+1} + \frac{5/4}{z-1} + \frac{2}{z}, & \Lambda_{+++ -}) \quad \omega &= \frac{5/4}{z+1} + \frac{5/4}{z-1} + \frac{-1}{z}, \\
 \Lambda_{++ - +}) \quad \omega &= \frac{5/4}{z+1} + \frac{-1/4}{z-1} + \frac{2}{z}, & \Lambda_{++ - -}) \quad \omega &= \frac{5/4}{z+1} + \frac{-1/4}{z-1} + \frac{-1}{z}, \\
 \Lambda_{+ - ++}) \quad \omega &= \frac{-1/4}{z+1} + \frac{5/4}{z-1} + \frac{2}{z}, & \Lambda_{+ - +-}) \quad \omega &= \frac{-1/4}{z+1} + \frac{5/4}{z-1} + \frac{-1}{z}, \\
 \Lambda_{+ - - +}) \quad \omega &= \frac{-1/4}{z+1} + \frac{-1/4}{z-1} + \frac{2}{z}, & \Lambda_{+ - - -}) \quad \omega &= \frac{-1/4}{z+1} + \frac{-1/4}{z-1} + \frac{-1}{z}.
 \end{aligned}$$

By step 3, there exists a monic polynomial of degree n satisfying relation (A.1):

$$\begin{aligned}
\Lambda_{++++}) \quad & \partial_z^2 \hat{P}_n + \frac{(3z+2)(3z-2)}{z(z+1)(z-1)} \partial_z \hat{P}_n + \frac{n(n+8)}{(z+1)(1-z)} \hat{P}_n = 0, \\
\Lambda_{+++-}) \quad & \partial_z^2 \hat{P}_n + \frac{3z^2+2}{z(z+1)(z-1)} \partial_z \hat{P}_n + \frac{n(n+2)}{(z+1)(1-z)} \hat{P}_n = 0, \\
\Lambda_{++-+}) \quad & \partial_z^2 \hat{P}_n + \frac{6z^2-3z-4}{z(z+1)(z-1)} \partial_z \hat{P}_n + \frac{nz(n+5)+6}{z(z+1)(1-z)} \hat{P}_n = 0, \\
\Lambda_{++--}) \quad & \partial_z^2 \hat{P}_n + \frac{3z-2}{z(z+1)(1-z)} \partial_z \hat{P}_n + \frac{nz(n-1)-12}{4z(z+1)(1-z)} \hat{P}_n = 0, \\
\Lambda_{+-++}) \quad & \partial_z^2 \hat{P}_n + \frac{6z^2+3z-4}{z(z+1)(z-1)} \partial_z \hat{P}_n + \frac{z(n^2+5)-6}{z(z+1)(1-z)} \hat{P}_n = 0, \\
\Lambda_{+--+}) \quad & \partial_z^2 \hat{P}_n + \frac{3z+2}{z(z+1)(z-1)} \partial_z \hat{P}_n + \frac{nz(n-1)+3}{z(z+1)(1-z)} \hat{P}_n = 0, \\
\Lambda_{+---}) \quad & \partial_z^2 \hat{P}_n + \frac{3z^2-4}{z(z+1)(z-1)} \partial_z \hat{P}_n + \frac{n(n+2)}{(z+1)(1-z)} \hat{P}_n = 0, \\
\Lambda_{+----}) \quad & \partial_z^2 \hat{P}_n + \frac{3z^2-2}{z(z+1)(1-z)} \partial_z \hat{P}_n + \frac{n(4-n)}{(z+1)(z-1)} \hat{P}_n = 0.
\end{aligned}$$

The polynomial P_n of degree n exists for $\lambda_n \in \Lambda_b$ with n even, that is, $\Lambda_n = \{n \in \mathbb{Z} : 16n^2 + 16n + 3\}$, for $\Lambda_{++-+})$ and $\Lambda_{+--+})$. Therefore $E = E_n = \{n \in \mathbb{Z} : -4n^2 - 4n\}$.

The corresponding solutions for Λ_n are

$$\begin{aligned}
\Lambda_{++++}) \quad & \Phi_{1,n} = \hat{P}_{2n} \hat{f}_n \Phi_{1,0}, \quad \Phi_{1,0} = \frac{\sqrt[4]{(z^2-1)^5}}{z} \quad \hat{f}_n = 1, \quad \hat{\Psi}_{1,0} = z - \frac{1}{z}, \\
\Lambda_{+--+}) \quad & \hat{\Phi}_{2,n} = \hat{Q}_{2n} \hat{f}_n \hat{\Phi}_{2,0}, \quad \hat{\Phi}_{2,0} = \frac{z^2}{\sqrt[4]{z^2-1}} \quad \hat{f}_n = 1 \quad \hat{\Psi}_{2,0} = \frac{z^2}{\sqrt{z^2-1}}.
\end{aligned}$$

These two solutions are equivalent to the same solution of the original Schrödinger equation and correspond to the well known supersymmetric quantum mechanics approach to this Pöschl-Teller potential, [16, 17]. Furthermore, for all $\lambda \in \Lambda_n$,

$$\begin{aligned}
\mathrm{DGal}(\hat{\mathbf{L}}_\lambda/\mathbb{C}(x)) &= \mathbb{G}^{[4]}, \quad \mathrm{DGal}(\hat{L}_\lambda/\hat{K}) = \mathrm{DGal}(L_\lambda/K) = e, \\
\dim_{\mathbb{C}} \mathcal{E}(\hat{\mathbf{H}} - \lambda) &= 2, \quad \dim_{\mathbb{C}} \mathcal{E}(\hat{H} - \lambda) = \dim_{\mathbb{C}} \mathcal{E}(H - \lambda) = 4.
\end{aligned}$$

6.4. Searching for Potentials From Parameterized Differential Equations. In order to search for new potentials using $\hat{\partial}_z$, our main object will be the family of differential equations presented by Darboux in [20], which can be written in the form

$$(6.10) \quad \partial_z^2 \hat{y} + \hat{P} \partial_z \hat{y} + (\hat{Q} - \lambda \hat{R}) \hat{y} = 0, \quad \hat{P}, \hat{Q}, \hat{R} \in \hat{K}.$$

We recall that some Riemann differential equations, presented previously, correspond to this kind.

When we have a differential equation in the form (6.10), we reduce it to put it in the form of the reduced algebrized Schrödinger equation $\hat{\mathbf{H}}\Phi = \lambda\Phi$, checking that $\mathrm{Card}(\Lambda) > 1$. Thus, starting with the potential $\hat{\mathbf{V}}$ and arriving to the potential V , we obtain the Schrödinger equation $H\Psi = \lambda\Psi$. This heuristic methodology is detailed below.

- (1) Reduce a differential equation of the form (6.10) and put it in the form $\widehat{\mathbf{H}}\Phi = \lambda\Phi$, checking that $\text{Card}(\Lambda) > 1$; to avoid triviality, α should be a non-constant function.
- (2) Write $\mathcal{W} = \frac{1}{4}\partial_z(\ln \alpha)$ and obtain $\widehat{V}(z) = \alpha(\widehat{\mathbf{V}} - \partial_z\mathcal{W} - \mathcal{W}^2)$.
- (3) Solve the differential equation $(\partial_x z)^2 = \alpha$ and write $z = z(x)$, $V(x) = \widehat{V}(z(x))$.

To illustrate this method, we present the following examples.

Bessel Potentials

- (From Darboux transformations over $V = 0$) In the differential equation

$$\partial_z^2 \Phi = \left(\frac{n(n+1)}{z^2} + \mu \right) \Phi, \quad \mu \in \mathbb{C},$$

we see that $\lambda = -n(n+1)$ and $\alpha = z^2$. Applying the method, we obtain $\widehat{\mathbf{V}} = \mu$ and $\widehat{V}(z) = \mu z^2 + \frac{1}{4}$ with $z = z(x) = e^{\pm x}$. Thus, we have obtained the potentials $V(x) = \widehat{V}(z(x)) = \mu e^{\pm 2x} + \frac{1}{4}$ (compare with [27, §6.9]).

- (From Bessel differential equation) The equation

$$\partial_z^2 y + \frac{1}{z} \partial_z y + \frac{z^2 - n^2}{z^2} y = 0, \quad n \in \frac{1}{2} + \mathbb{Z},$$

is transformed into the reduced equation

$$\partial_z^2 \Phi = \left(\frac{n^2}{z^2} - \frac{4z^2 + 1}{4z^2} \right) \Phi.$$

We can see that $\lambda = -n^2$, $\alpha = z^2$, obtaining $\widehat{\mathbf{V}} = -z^2 - \frac{1}{4}$, $\widehat{V} = -z^4 - \frac{1}{4}z^2 + \frac{1}{4}$ and $z = z(x) = e^{\pm x}$. Thus, we have obtained the potential $V(x) = \widehat{V}(z(x)) = -e^{\pm 4x} - \frac{1}{4}e^{\pm 2x} + \frac{1}{4}$ (compare with [27, §6.9]).

We remark that the previous examples give us potentials related with the Morse potential, due to their solutions are given in term of Bessel functions.

We can apply this method to equations such as Whittaker, Hypergeometric and in particular, differential equations involving orthogonal polynomials (compare with [16, §5]).

Final Remarks

In this paper we gave, in contemporary terms, a formalization of original ideas and intuitions given by G. Darboux, E. Witten and L. É. Gendenshtein in the context of the Galois theory of linear differential equations. We found the following facts.

- The superpotential is an algebraic solution of the Riccati equation associated with a potential, defined over a differential field.
- The Darboux transformation was interpreted as an isogaloisian transformation, allowing to obtain isomorphisms between their eigenrings.

- We introduced the Hamiltonian algebrization method, which in particular allows to apply algorithmic tools such as Kovacic's algorithm to obtain the solutions, differential Galois groups and eigenring s of second order linear differential equations. We applied successfully this algebrization procedure to solve problems in Supersymmetric quantum mechanics.
- We can construct algebraically solvable and non-trivial algebraically quasi-solvable potentials in the following ways.
 - (1) Giving a potential where, for $\lambda = \lambda_0$, the Schrödinger equation is integrable. After we put $\lambda \neq \lambda_0$, we check that the Schrödinger equation is integrable for more than one value of the parameter λ .
 - (2) Giving a superpotential to obtain the potential, after which we check whether the Schrödinger equation is integrable for more than one value of the parameter λ .
 - (3) Using parameterized second order linear differential equations and applying an inverse process in the Hamiltonian algebrization method. In particular, we can use algebraically solvable and algebraically quasi-solvable potentials, special functions with parameters (for example with polynomial solutions).

This paper is a starting point to analyze quantum theories through Galoisian theories. Therefore open questions and future work arise in a natural way: supersymmetric quantum mechanics with dimension greater than 2, relationship between algebraic and analytic spectrums, etc.

As a conclusion, as happen in other areas of the field of differential equations, in view of the many families of examples studied along this paper, we can conclude that the *differential Galois theory* is a natural framework in which some aspects of *supersymmetric quantum mechanics* may appear more clearly.

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Appendix A. Kovacic's Algorithm

Kovacic in 1986 (see [36]) introduced an algorithm to solve the differential equation $\partial_x^2 \zeta = r\zeta$, where $r \in \mathbb{C}(x)$, see also [4, 24, 25, 62]. We recall this algorithm to set the notations.

Each case in Kovacic's algorithm is related with each one of the algebraic subgroups of $SL(2, \mathbb{C})$ and the associated Riccati equation

$$\partial_x v = r - v^2 = (\sqrt{r} - v)(\sqrt{r} + v), \quad v = \frac{\partial_x \zeta}{\zeta}.$$

There are four cases in Kovacic's algorithm. Case 1 concerns the case when the Riccati equation has a rational solution (or more); cases 2 and 3 concern cases when the Riccati equation admits an algebraic solution; case 4 is the non-integrable case.

Our differential equation is given by $\partial_x^2 \zeta = r\zeta$ where $r = \frac{s}{t}$, $s, t \in \mathbb{C}[x]$. We use the following notations:

- (1) Γ' is the set of (finite) poles of r : $\Gamma' = \{c \in \mathbb{C} : t(c) = 0\}$.
- (2) $\Gamma = \Gamma' \cup \{\infty\}$.
- (3) The order $\circ(r_c)$ of r at $c \in \Gamma'$ is the multiplicity of c as a pole of r .
- (4) The order $\circ(r_\infty)$ of r at ∞ is the order of ∞ as a zero of r ,
i.e $\circ(r_\infty) = \deg(t) - \deg(s)$.

Case 1. Let $[\sqrt{r}]_c$ (resp. $[\sqrt{r}]_\infty$) represent the Laurent series (if defined) of \sqrt{r} at c (resp. infinity). For $p \in \Gamma$, we will define $\varepsilon(p) \in \{+, -\}$ below. Finally, the complex numbers $\alpha_c^+, \alpha_c^-, \alpha_\infty^+, \alpha_\infty^-$ will be defined in the first step. If the differential equation has no poles it only can fall in this case.

Step 1. For each $c \in \Gamma$:

(c_0): If $\circ(r_c) = 0$, then

$$[\sqrt{r}]_c = 0, \quad \alpha_c^\pm = 0.$$

(c_1): If $\circ(r_c) = 1$, then

$$[\sqrt{r}]_c = 0, \quad \alpha_c^\pm = 1.$$

(c_2): If $\circ(r_c) = 2$, and

$$r = b(x - c)^{-2} + \dots, \quad \text{then}$$

$$[\sqrt{r}]_c = 0, \quad \alpha_c^\pm = \frac{1 \pm \sqrt{1 + 4b}}{2}.$$

(c_3): If $\circ(r_c) = 2v \geq 4$, and

$$r = (a(x - c)^{-v} + \dots + d(x - c)^{-2})^2 + b(x - c)^{-(v+1)} + \dots, \quad \text{then}$$

$$[\sqrt{r}]_c = a(x - c)^{-v} + \dots + d(x - c)^{-2}, \quad \alpha_c^\pm = \frac{1}{2} \left(\pm \frac{b}{a} + v \right).$$

(∞_1): If $\circ(r_\infty) > 2$, then

$$[\sqrt{r}]_\infty = 0, \quad \alpha_\infty^+ = 0, \quad \alpha_\infty^- = 1.$$

(∞_2): If $\circ(r_\infty) = 2$, and $r = \dots + bx^2 + \dots$, then

$$[\sqrt{r}]_\infty = 0, \quad \alpha_\infty^\pm = \frac{1 \pm \sqrt{1 + 4b}}{2}.$$

(∞_3): If $\circ(r_\infty) = -2v \leq 0$, and

$$r = (ax^v + \dots + d)^2 + bx^{v-1} + \dots, \quad \text{then}$$

$$[\sqrt{r}]_\infty = ax^v + \dots + d, \quad \text{and} \quad \alpha_\infty^\pm = \frac{1}{2} \left(\pm \frac{b}{a} - v \right).$$

Step 2. Find $D \neq \emptyset$ defined by

$$D = \left\{ n \in \mathbb{Z}_+ : n = \alpha_\infty^{\varepsilon(\infty)} - \sum_{c \in \Gamma'} \alpha_c^{\varepsilon(c)}, \forall (\varepsilon(p))_{p \in \Gamma} \right\}.$$

If $D = \emptyset$, then we should start with the case 2. Now, if $\text{Card}(D) > 0$, then for each $n \in D$ we search $\omega \in \mathbb{C}(x)$ such that

$$\omega = \varepsilon(\infty) [\sqrt{r}]_\infty + \sum_{c \in \Gamma'} \left(\varepsilon(c) [\sqrt{r}]_c + \alpha_c^{\varepsilon(c)} (x - c)^{-1} \right).$$

Step 3. For each $n \in D$, search for a monic polynomial P_n of degree n with

$$(A.1) \quad \partial_x^2 P_n + 2\omega \partial_x P_n + (\partial_x \omega + \omega^2 - r) P_n = 0.$$

If success is achieved then $\zeta_1 = P_n e^{\int \omega}$ is a solution of the differential equation. Else, case 1 cannot hold.

Case 2.

Step 1. For each $c \in \Gamma'$ and for ∞ , we define the sets $E_c \subset \mathbb{Z}$ and $E_\infty \subset \mathbb{Z}$ as follows:

- (c_1): If $\circ(r_c) = 1$, then $E_c = \{4\}$
 (c_2): If $\circ(r_c) = 2$, and $r = \dots + b(x-c)^{-2} + \dots$, then

$$E_c = \left\{ 2 + k\sqrt{1+4b} : k = 0, \pm 2 \right\}.$$

- (c_3): If $\circ(r_c) = v > 2$, then $E_c = \{v\}$
 (∞_1): If $\circ(r_\infty) > 2$, then $E_\infty = \{0, 2, 4\}$
 (∞_2): If $\circ(r_\infty) = 2$, and $r = \dots + bx^2 + \dots$, then

$$E_\infty = \left\{ 2 + k\sqrt{1+4b} : k = 0, \pm 2 \right\}.$$

- (∞_3): If $\circ(r_\infty) = v < 2$, then $E_\infty = \{v\}$

If some $E_c = \emptyset$ then switch to next case.

Step 2. Compute D defined by

$$D = \left\{ n \in \mathbb{Z}_+ : n = \frac{1}{2} \left(e_\infty - \sum_{c \in \Gamma'} e_c \right), \forall e_p \in E_p, \quad p \in \Gamma \right\}.$$

If $D = \emptyset$, then switch to case 3. If $\text{Card}(D) > 0$, then for each $n \in D$, compute the corresponding rational function θ defined by

$$\theta = \frac{1}{2} \sum_{c \in \Gamma'} \frac{e_c}{x-c}.$$

Step 3. For each $n \in D$, search (via linear algebra) for a monic polynomial P_n of degree n , such that

$$(A.2) \quad \partial_x^3 P_n + 3\theta \partial_x^2 P_n + (3\partial_x \theta + 3\theta^2 - 4r) \partial_x P_n + (\partial_x 2\theta + 3\theta \partial_x \theta + \theta^3 - 4r\theta - 2\partial_x r) P_n = 0.$$

If P_n does not exist, then case 2 does not hold. If such a polynomial P_n is found, set $\phi = \theta + \partial_x P_n / P_n$ and let ω be a solution of

$$\omega^2 + \phi \omega + \frac{1}{2} (\partial_x \phi + \phi^2 - 2r) = 0.$$

Then $\zeta_1 = e^{\int \omega}$ is a solution of the differential equation.

Case 3.

Step 1. For each $c \in \Gamma'$ and for ∞ we define the sets $E_c \subset \mathbb{Z}$ and $E_\infty \subset \mathbb{Z}$ as follows:

- (c_1): If $\circ(r_c) = 1$, then $E_c = \{12\}$

(c_2): If $\circ(r_c) = 2$, and $r = b(x - c)^{-2} + \dots$, then

$$E_c = \left\{ 6 + k\sqrt{1 + 4b} : k = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6 \right\}.$$

(∞): If $\circ(r_\infty) = v \geq 2$, and $r = \dots + bx^2 + \dots$, then

$$E_\infty = \left\{ 6 + \frac{12k}{m}\sqrt{1 + 4b} : k = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6 \right\}, \quad m \in \{4, 6, 12\}.$$

Step 2. For $m \in \{4, 6, 12\}$ successively, compute the set D defined by

$$D = \left\{ n \in \mathbb{Z}_+ : n = \frac{m}{12} \left(e_\infty - \sum_{c \in \Gamma'} e_c \right), \forall e_p \in E_p, \quad p \in \Gamma \right\}.$$

If $\text{Card}(D) > 0$, then for each $n \in D$ with its respective m , compute the rational function

$$\theta = \frac{m}{12} \sum_{c \in \Gamma'} \frac{e_c}{x - c}$$

and the (denominator) polynomial S defined as $S = \prod_{c \in \Gamma'} (x - c)$.

Step 3. For each $n \in D$, with its respective m , compute a monic polynomial $P_n = P$ of degree n , such that its coefficients can be determined recursively by

$$P_{-1} = 0, \quad P_m = -P,$$

$$P_{i-1} = -S\partial_x P_i - ((m-i)\partial_x S - S\theta)P_i - (m-i)(i+1)S^2 r P_{i+1},$$

where $i \in \{0, 1, \dots, m-1, m\}$. If P does not exist, then the differential equation is not integrable because it falls in Case 4.

If P exists, a solution of the differential equation is given by $\zeta = e^{\int \omega}$, where ω is a zero of the polynomial of degree m

$$\sum_{i=0}^m \frac{S^i P_i}{(m-i)!} \omega^i = 0$$

Case 4. In none of the above holds, then the equation is non-integrable and its Galois group is $SL(2)$.

Remark A.1 ([4]). If the differential equation falls only in the case 1 of Kovacic's algorithm, then its differential Galois group is given by one of the following groups:

I1: e when the algorithm provides two rational solutions.

I2: $\mathbb{G}^{[n]}$ when the algorithm provides two algebraic solutions ζ_1, ζ_2 such that $\zeta_1^n, \zeta_2^n \in \mathbb{C}(x)$ and $\zeta_1^{n-1}, \zeta_2^{n-1} \notin \mathbb{C}(x)$.

I3: $\mathbb{G}^{\{n\}}$ when the algorithm provides only one algebraic solution ζ such that $\zeta^n \in \mathbb{C}(x)$ with n minimal.

I4: \mathbb{G}_m when the algorithm provides two non-algebraic solutions.

I5: \mathbb{G}_a when the algorithm provides one rational solution and the second solution is not algebraic.

I6: \mathbb{B} when the algorithm only provides one solution ζ such that ζ and its square are not rational functions.

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